

THE BOUNDARY AT INFINITY OF A ROUGH CAT(0) SPACE

S.M. BUCKLEY AND K. FALK

ABSTRACT. We develop the boundary theory of rough CAT(0) spaces, a class of spaces that contains both Gromov hyperbolic and CAT(0) spaces. The resulting theory generalizes the common features of the Gromov boundary of a Gromov hyperbolic space and the ideal boundary of a complete CAT(0) space. It is not assumed that the spaces are geodesic or proper.

1. INTRODUCTION

The boundary theory of Gromov hyperbolic and complete CAT(0) spaces share common features; by “boundary”, we always mean some sort of boundary at infinity. In particular if X is both a Gromov hyperbolic space and a complete CAT(0) space, then it is well known that its Gromov boundary $\partial_G X$ and its ideal boundary $\partial_I X$ can naturally be identified. Furthermore under this identification, the canonical topology τ_G generated by the canonical gauge of metrics on the Gromov boundary equals the cone topology τ_C on the ideal boundary. See Section 2 for relevant definitions and references.

However it would be preferable to reconcile the common features of these two theories inside a larger class rather than in the intersection of the two classes. With a view to doing this, we defined a class of *rough CAT(0) spaces* (abbreviated rCAT(0)) in [3], where we also investigated the interior (i.e. non-boundary) geometry of such spaces. This new class is arguably the smallest natural class of spaces that properly contains all Gromov hyperbolic and all CAT(0) spaces; it is not assumed that the spaces involved are geodesic, proper, or even complete. Rough CAT(0) is closely related to the class of bolic spaces of Kasparov and Skandalis [11], [12] that was introduced in the context of their work on the Baum-Connes and Novikov Conjectures, and is also related to Gromov’s class of CAT(-1, ϵ) spaces [9], [7].

Building on [3], we investigate the boundary theory of rCAT(0) spaces in this paper. Unlike complete CAT(0) spaces, geodesic rays in an rCAT(0) space do not form the basis of a nice boundary theory, and completeness is not a useful assumption. Instead we replace geodesic rays by bouquets of short paths whose lengths tend to infinity; one version of these bouquets is closely related to the *roads* that Väisälä [13] introduced in the context of Gromov hyperbolic spaces. We then define what we call the *bouquet boundary* $\partial_B X$ of X , and the associated bordification $\overline{X}_B := X \cup \partial_B X$. Moreover we

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define a *bouquet topology* τ_B on \overline{X}_B , denoting the corresponding subspace topology on $\partial_B X$ also by τ_B . Similarly, we write $\overline{X}_I = X \cup \partial_I X$ and $\overline{X}_G = X \cup \partial_G X$.

The following pair of results show that the bouquet boundary with its associated topology is indeed the desired type of generalization.

Theorem 1.1. *Suppose X is a complete $CAT(0)$ space. Then \overline{X}_I equipped with the cone topology and \overline{X}_B equipped with the bouquet topology are naturally homeomorphic.*

Theorem 1.2. *Suppose X is a δ -hyperbolic space, $\delta \geq 0$. Then \overline{X}_B equipped with the bouquet topology and \overline{X}_G equipped with the canonical topology are naturally homeomorphic.*

The rest of this paper is organized as follows. After some preliminaries in Section 2, Section 3 reviews the parts of the basic theory of $rCAT(0)$ spaces developed in [3] that are needed here.

In Section 4, we investigate several definitions of the bouquet boundary, all defined using equivalence classes of bouquets of paths, and prove their equivalence as sets, i.e. there is a natural bijection between any pair of them. We also relate the bouquet, ideal, and end boundaries, and prove the following result.

Theorem 1.3. *If X is an unbounded proper $rCAT(0)$ space, then $\partial_B X$ is nonempty.*

Other possible definitions of the bouquet boundary use equivalence classes of points “tending to infinity” (but not in the sense typically employed for Gromov hyperbolic spaces). In Section 5, we prove that some (but not all!) of these definitions are equivalent as sets to the definitions in terms of path bouquets. We also show that in a Gromov hyperbolic space, all our sequential variants are equivalent as sets to the Gromov boundary.

Finally in Section 6, we define and investigate the bouquet topology τ_B , and prove the topological parts of the above results, as well as the following result.

Theorem 1.4. *If X is $rCAT(0)$, then \overline{X}_B is Hausdorff and first countable. If additionally X is proper then both \overline{X}_B and $\partial_B X$ are compact.*

2. PRELIMINARIES

Throughout this section, we suppose (X, d) is a metric space. We say that X is *proper* if every closed ball in X is compact.

We write $A \wedge B$ and $A \vee B$ for the minimum and maximum, respectively, of two numbers A, B .

We define a *h -short segment* from x to y , where $x, y \in X$, to be a path of length at most $d(x, y) + h$, $h \geq 0$. A *geodesic segment* is a 0-short segment. X is a *length space* if there is a h -short segment between each pair $x, y \in X$ for every $h > 0$, and X is a *geodesic space* if there is a geodesic segment between each pair $x, y \in X$.

A *geodesic ray* in X is a path $\gamma : [0, \infty) \rightarrow X$ such that each initial segment $\gamma|_{[0, t]}$ of γ is a geodesic segment. The *ideal boundary* $\partial_I X$ of X is the set of equivalence classes of geodesic rays in X , where two geodesic rays γ_1, γ_2 are said to be equivalent if $d(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t))$ is uniformly bounded for all $t \geq 0$, where $\tilde{\gamma}_i$ is the unit speed reparametrization of γ_i , $i = 1, 2$.

We refer the reader to [1, Part II] for the theory of CAT(0) spaces. The ideal boundary $\partial_1 X$ of a complete CAT(0) space can be identified with the set of geodesic rays from any fixed origin $o \in X$ [1, II.8.2].

Definition 2.1. The *cone topology* τ_C on the *ideal bordification* $\overline{X}_I := X \cup \partial_1 X$ of a complete CAT(0) space X is the topology with the following neighborhood basis:

$$\mathcal{B}(x) = \begin{cases} \{B(x, r) \mid r > 0\}, & x \in X, \\ \{U(x, r, t) \mid r, t > 0\}, & x \in \partial_B X, \end{cases}$$

where

$$B(x, r) = \{y \in X \mid d(y, x) < r\}$$

$$U(x, r, t) = \{y \in \overline{X}_I \setminus B(o, r) \mid d(p_t(x), p_t(y)) < r\},$$

and $p_t : \overline{X}_I \setminus B(o, r) \rightarrow X$ is the projection defined by the intersection of the metric sphere $S(o, r)$ and the geodesic segment or ray from o to x .

We refer the reader to [8], [6], [13], or [1, Part III.H] for the theory of Gromov hyperbolic spaces. We use the non-geodesic definition: a metric space (X, d) is *δ -hyperbolic*, $\delta \geq 0$, if

$$\langle x, z; w \rangle \geq \langle x, y; w \rangle \wedge \langle y, z; w \rangle - \delta, \quad x, y, z, w \in X,$$

where $\langle x, z; w \rangle$ is the Gromov product¹ defined by

$$2 \langle x, y; w \rangle = d(x, w) + d(y, w) - d(x, y).$$

Gromov sequences and the Gromov boundary have mainly been considered in Gromov hyperbolic spaces, but they have also been defined in general metric spaces [5].

Definition 2.2. A *Gromov sequence* in a metric space X is a sequence (x_n) in X such that $\langle x_m, x_n; o \rangle \rightarrow \infty$ as $m, n \rightarrow \infty$. If $x = (x_n)$ and $y = (y_n)$ are two such sequences, we write $(x, y) \in E$ if $\langle x_m, y_n; o \rangle \rightarrow \infty$ as $m, n \rightarrow \infty$. Then E is a reflexive symmetric relation on the set of Gromov sequences in X , so its transitive closure \sim is an equivalence relation on the set of Gromov sequences in X . Note that E is already an equivalence relation if X is Gromov hyperbolic, but this is not true in general metric spaces [5, 1.5]. The *Gromov boundary* $\partial_G X$ is the set of equivalence classes $[(x_n)]$ of Gromov sequences.

To simplify the statement of the following definition, we identify $x \in X$ with the singleton equivalence class $[(x_n)]$, where $x_n = x$ for all n .

Definition 2.3. The *Gromov bordification* $\overline{X}_G := X \cup \partial_G X$ of a Gromov hyperbolic space X can be equipped with the *canonical topology* τ_G that has the following neighborhood basis:

$$\mathcal{G}(x) = \begin{cases} \{B(x, r) \mid r > 0\}, & x \in X, \\ \{V(x, r) \mid r > 0\}, & x \in \partial_G X, \end{cases}$$

where $B(x, r)$ is as in Definition 2.1 and

$$V(x, r) = \{y \in \overline{X}_G \mid \exists \text{ Gromov sequences } (a_n), (b_n) : \\ [(a_n)] = x, [(b_n)] = y, \text{ and } \liminf_{m, n \rightarrow \infty} \langle a_m, b_n; o \rangle > r\},$$

¹ $\langle x, y; w \rangle$ is more commonly written as $\langle x, y \rangle_w$. Our notation is designed to avoid double subscripts.

The topology τ_G is often given only for $\partial_G X$ where it is associated with a canonical gauge of metrics, but we do not need these metrics. However τ_G defined on all of \bar{X}_G can be found in the literature: for instance, τ_G is equivalent to the topology of [1, III.H.3.5] (for proper geodesic hyperbolic spaces, to that of [10, p. 6], and to the topology \mathcal{T}_1^* in [13, 5.29] (but it is coarser than \mathcal{T}^* also defined there).

3. ROUGH CAT(0) SPACES

In this section we review various notions of rough CAT(0) introduced in our first paper [3], as well as some rCAT(0) results that we need here. Except where otherwise referenced or proved, proofs of statements in this section can be found in [3], where the reader can also find a more detailed discussion of the concepts introduced below.

For the following definitions of short triangles and comparison points, we denote h -short segments connecting points $x, y \in X$ by $[x, y]_h$. We use the notation $[x, y]_h$ also for the image of this path, so instead of $z = \gamma(t)$ for some $0 \leq t \leq L$, we write $z \in [x, y]_h$. Given such a path γ and point $z = \gamma(t)$, we denote by $[x, z]_h$ and $[z, y]_h$ the subpaths $\gamma|_{[0, t]}$ and $\gamma|_{[t, L]}$, respectively, both of which are also h -short segments. This notation is ambiguous: given points x, y in a length space X with at least two points, there are always many short segments $[x, y]_h$ for each $h > 0$. However the choice of $[x, y]_h$, once made, does not affect the truth of the underlying statement.

A h -short triangle $T := T_h(x_1, x_2, x_3)$ with vertices $x_1, x_2, x_3 \in X$ is defined as a collection of h -short segments $[x_1, x_2]_h$, $[x_2, x_3]_h$ and $[x_3, x_1]_h$, and a *comparison triangle* is then a geodesic triangle $\bar{T} := T(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the model space, Euclidean \mathbb{R}^2 , so that $|\bar{x}_i - \bar{x}_j| = d(x_i, x_j)$, $i, j \in \{1, 2, 3\}$. We say that $\bar{u} \in \bar{T}$ is a h -comparison point for $u \in T$, say $u \in [x_1, x_2]_h$, if

$$|\bar{x} - \bar{u}| \leq \text{len}([x, u]_h) \quad \text{and} \quad |\bar{u} - \bar{y}| \leq \text{len}([u, y]_h).$$

Note that \bar{u} is not uniquely determined by u , but we do have

$$|\bar{x} - \bar{u}| \geq \text{len}([x, u]_h) - h \quad \text{and} \quad |\bar{u} - \bar{y}| \geq \text{len}([u, y]_h) - h.$$

Given a h -short triangle $T := T_h(x, y, z)$ in any length space X , and $u \in T$, we can always find a comparison triangle and h -comparison point in \mathbb{R}^2 .

Let $C \geq 0$, and $h \geq 0$. Suppose $T_h(x, y, z)$ is a h -short triangle in X . We say that $T_h(x, y, z)$ satisfies the C -rough CAT(0) condition if given a comparison triangle $T(\bar{x}, \bar{y}, \bar{z})$ in \mathbb{R}^2 associated with $T_h(x, y, z)$, we have

$$d(u, v) \leq |\bar{u} - \bar{v}| + C,$$

whenever u, v lie on different sides of $T_h(x, y, z)$ and $\bar{u}, \bar{v} \in T(\bar{x}, \bar{y}, \bar{z})$ are corresponding h -comparison points.

Definition 3.1. We say that (X, d) is C -rCAT(0), $C > 0$, if $T_h(x, y, z)$ satisfies the C -rough CAT(0) condition whenever $T_h(x, y, z)$ is a h -short triangle in X with

$$(3.2) \quad h \leq H(x, y, z) = \frac{1}{1 \vee d(x, y) \vee d(x, z) \vee d(y, z)}.$$

We omit the *roughness constant* C in the above notation if its value is unimportant.

Our specific choice of H , although often useful, seems somewhat contrived. A more natural definition would be to assume that there exists some $H : X \times X \times X \rightarrow (0, \infty)$ such that the C -rCAT(0) condition holds for $T_h(x, y, z)$ whenever $h \leq H(x, y, z)$. We

call this the C - $rCAT(0;*)$ condition, $C > 0$. It is formally weaker than the C - $rCAT(0)$ condition, but the two definitions are equivalent in the sense that a C - $rCAT(0;*)$ space is C' - $rCAT(0)$, with $C' = 3C + 2 + \sqrt{3}$. Outside of esthetics, another advantage of the C - $rCAT(0;*)$ condition is that it is an interesting condition for C near 0, unlike C - $rCAT(0)$; see Proposition 3.3.

To ensure that CAT(0) spaces (or even just the Euclidean plane) are $rCAT(0)$ spaces, we need h to be bounded by at most a fixed multiple of the above function H ; see [3, Example 3.3]. In particular, one cannot pick a constant bound for h . Combining Theorem 4.5 and Corollary 4.6 of [3], we do however get the following result.

Proposition 3.3. *A CAT(0) space X is C - $rCAT(0)$ with $C = 2 + \sqrt{3}$, and C - $rCAT(0;*)$ for all $C > 0$.*

The analogous relationship with Gromov hyperbolic spaces is given by the following result which follows from the proof of [3, Theorem 3.18].

Proposition 3.4. *A δ -hyperbolic space, $\delta \geq 0$, is C - $rCAT(0)$ with $C = 2 + 4\delta$.*

The CAT(0) condition is equivalent to a weaker version of itself where the comparison inequality is assumed only when one point is a vertex, and one can even restrict the other point to being the midpoint of the opposite side. Analogously *weak* and *very weak* C - $rCAT(0)$ spaces are defined by making the corresponding changes to the above definitions of C - $rCAT(0)$ spaces. Trivially an $rCAT(0)$ space is weak $rCAT(0)$, and a weak $rCAT(0)$ space is very weak $rCAT(0)$, but we cannot at this time determine the truth of the reverse implications.

We will not use the weak and very weak $rCAT(0)$ variants in this paper, but let us mention that the very weak variant is quantitatively equivalent to the notion of bolicity introduced by Kasparov and Skandalis [11], [12]; see [3, Proposition 3.11].

The weak C - $rCAT(0)$ condition can be written in the following more explicit form: if $u = \lambda(s)$, where $\lambda : [0, L] \rightarrow X$ is a h -short path from y to z parametrized by arclength, h satisfies the usual bound, and $0 \leq t \leq 1$ is such that $td(y, z) \leq s$ and $(1 - t)d(y, z) \leq L - s$, then

$$(3.5) \quad (d(x, u) - C)^2 \leq (1 - t)(d(x, y))^2 + t(d(x, z))^2 - t(1 - t)(d(y, z))^2.$$

This inequality holds *a fortiori* in C - $rCAT(0)$ spaces, a fact that will be useful later. Note that (3.5) follows easily from the definition of weak $rCAT(0)$ and the following easily proved equality in the Euclidean plane for a triangle with vertices x, y, z and a point u on the side yz such that $|y - u| = t|y - z|$:

$$|x - u|^2 = (1 - t)|x - y|^2 + t|x - z|^2 - t(1 - t)|y - z|^2.$$

We have the following rough convexity lemma for $rCAT(0)$ spaces.

Lemma 3.6. *Suppose a_1, a_2, b_1, b_2 are points in a C - $rCAT(0)$ space X . Let $\gamma_i : [0, 1] \rightarrow X$ be constant speed h_i -short paths from a_i to b_i , $i = 1, 2$, where $h_i \leq 1/(1 \vee d(a_i, b_i))$. Then there exists a constant C' such that*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1 - t)d(a_1, a_2) + td(b_1, b_2) + C'.$$

In fact we can take $C' = 2C$, and if either $a_1 = a_2$ or $b_1 = b_2$, we can take $C' = C$. If X is CAT(0), we can take C' to be any positive number if we add the restriction that $h_1, h_2 \leq \epsilon$, where $\epsilon = \epsilon(C', d(a_1, b_1) \vee d(a_2, b_2)) > 0$ is sufficiently small.

Except for its last statement, the above lemma is just a restatement of [3, Lemma 4.7]. The last statement follows from the corresponding convexity result for geodesic segments in a CAT(0) space (which states that the estimate of Lemma 3.6 holds with $C' = 0$) and the fact that a h -short path between any fixed pair of points x, y in a CAT(0) space is forced to stay arbitrarily close to the geodesic segment between these points as long as both h and $hd(x, y)$ are sufficiently small [3, Theorem 4.5].

Lastly we state and prove two lemmas that we will need in Sections 4 and 6.

Lemma 3.7. *Suppose o_1, o_2, u_1, u_2, x are points in a C -rCAT(0) space X such that:*

- (a) *For $i = 1, 2$, u_i lies on a path γ_i of length L_i from o_i to x ;*
- (b) *For $i = 1, 2$, $d(o_i, u_i) \leq d(o_i, x)$;*
- (c) *$d(o_1, u_1) = d(o_2, u_2)$;*
- (d) *For $i = 1, 2$, γ_i is h -short, where $h := H(o_1, o_2, x)$ and H is as defined in (3.2).*

Then $d(u_1, u_2) \leq C + d(o_1, o_2)$.

Proof. Let $T := T(o_1, o_2, x)$ be a h -short triangle such that γ_1, γ_2 are two of its sides, and let $\bar{T} := T(\bar{o}_1, \bar{o}_2, \bar{x})$ be a comparison triangle. Write $\delta := d(o_1, u_1) = d(o_2, u_2)$. For $i = 1, 2$, let \bar{u}_i be the point on $[\bar{o}_i, \bar{x}]$ with $|\bar{o}_i - \bar{u}_i| = \delta$. We claim that $|\bar{u}_1 - \bar{u}_2| \leq |\bar{o}_1 - \bar{o}_2|$. Since it is readily verified that \bar{u}_i is a h -comparison point for u_i , the desired conclusion follows by applying the C -rCAT(0) condition to this claimed inequality.

If the sidelengths a, b, c of a Euclidean triangle $T(t)$ are changing with time t in such a way that $b'(t) = c'(t) = 1$, and if the angle A opposite the side of length a is constant, then differentiating the cosine rule gives

$$2aa' = 2b + 2c - 2(b + c) \cos A$$

which immediately gives $a'(t) \geq 0$. Applying this fact with b increasing from $|\bar{u}_1 - \bar{x}|$ to $|\bar{o}_1 - \bar{x}|$, c increasing from $|\bar{u}_2 - \bar{x}|$ to $|\bar{o}_2 - \bar{x}|$, and a changing from $|\bar{u}_1 - \bar{u}_2|$ to $|\bar{o}_1 - \bar{o}_2|$, the claim follows. \square

Remark 3.8. If we replace assumption (c) with the assumption $\text{len}(\lambda_1) = \text{len}(\lambda_2)$, where λ_i is the subpath of γ_i from o_i to u_i , then we can take as comparison points the points \bar{u}_i on $[o_i, u_i]$ for which $|\bar{o}_i - \bar{u}_i| = \text{len}(\lambda_i)$, $i = 1, 2$. The conclusion of Lemma 3.7 now follows in the same manner.

Lemma 3.9. *Suppose o, u_1, u_2, x_1, x_2 are points in a C -rCAT(0) space X and that:*

- (a) *there exists $s \geq 0$ such that for $i = 1, 2$, $u_i = \gamma_i(s)$ for some unit speed path γ_i of length L_i from o to x_i ;*
- (b) *for $i = 1, 2$, γ_i is h -short, where $h := H(o, x_1, x_2)$ and H is as defined in (3.2).*

Then $d(u_1, u_2) \leq C + d(x_1, x_2)$.

Proof. As for Lemma 3.7, the proof reduces to an estimate for planar triangles. Specifically, we claim that if $T = T(o, x_1, x_2)$ is a triangle in the Euclidean plane and if $u_i \in [o, x_i]$ with $|o - u_i| = s$ for $i = 1, 2$, then $|u_1 - u_2| \leq |x_1 - x_2|$. By symmetry, it suffices to establish this claim for the case $|o - x_1| \leq |o - x_2|$. Since $|u_1 - u_2|$ is a linear function of s , we may as well assume that $u_1 = x_1$. By considering a triangle $T(t)$ as in the proof of Lemma 3.7, this again follows by calculus. \square

4. BOUQUET CONSTRUCTIONS

In this section we first introduce the various concepts required to define several variant *bouquet boundaries* of an rCAT(0) space X . We then show that all of these notions can be identified in a natural way with each other. Next we explore the relationship between the bouquet boundary and the ideal boundary, showing that they can be naturally identified in a complete CAT(0) space. Finally, we explore the relationship between ends and the bouquet boundary, and prove Theorem 1.3.

As motivation for the bouquet boundary, suppose $\gamma : [0, \infty) \rightarrow X$ is a geodesic ray parametrized by arclength in an rCAT(0) space X , with $\gamma(0) = o$. One of the basic properties of a complete CAT(0) space that we would like to emulate is that if $o' \in X$ is any other point, then there is a unit speed geodesic ray $\gamma' : [0, \infty) \rightarrow X$ with $\gamma'(0) = o'$ and $\sup_{t \geq 0} d(\gamma(t), \gamma'(t)) < \infty$. The standard proof of this involves taking a sequence of geodesic segments from o' to $\gamma(t_n)$ where $t_n \uparrow \infty$. The resulting unit speed paths $\gamma_n : [0, L_n] \rightarrow X$ are such that $d(\gamma_m(t), \gamma_n(t))$ is uniformly bounded for all $m, n \in \mathbb{N}$ and all $0 \leq t \leq L_m \wedge L_n$. Moreover if we fix t and pick $m, n \geq n_0$, then this uniform bound on $d(\gamma_m(t), \gamma_n(t))$ tends to 0, and $L_m \wedge L_n \rightarrow \infty$ as $n_0 \rightarrow \infty$. Defining $\gamma'(t) = \lim_{n \rightarrow \infty} \gamma_n(t)$ for all $t \geq 0$ gives a geodesic ray γ' from o' .

If X is merely rCAT(0) and if we use h_n -short paths γ_n for some appropriately small positive numbers h_n , then we can similarly derive a uniform bound on $d(\gamma_m(t), \gamma_n(t))$. However the rCAT(0) condition does not imply that this bound tends to 0 for $m, n \geq n_0 \rightarrow \infty$, so completeness is of no use. To overcome this obstacle, we discard geodesic rays and instead construct a boundary using sequences of paths such as (γ_n) above. The key features of (γ_n) are that all segments γ_n have a common origin, their lengths are increasing and tending to infinity (this may require that we take a subsequence above), and $d(\gamma_m(t), \gamma_n(t))$ is uniformly bounded whenever it is defined.

Bouquets (γ_n) with a uniform bound on $d(\gamma_m(t), \gamma_n(t))$ are the most natural concept arising from the above considerations, and are closely related to the *roads* that Väisälä [13] introduced in the context of Gromov hyperbolic spaces. There are two useful variants of this concept that lead to a naturally equivalent bouquet boundary. The first is a loose bouquet, where the bound on $d(\gamma_m(t), \gamma_n(t))$ is not uniform, but is instead allowed to grow more slowly than the smaller of the two distances $d(o', \gamma_m(t))$ and $d(o', \gamma_n(t))$; such loose bouquets are needed in the next section to investigate sequential versions of the bouquet boundary. The second equivalent notion is a standard bouquet, a tighter notion than a bouquet which is needed to define the bouquet topology in Section 6.

4.A. Bouquets: Definitions and basics.

Definition 4.1. A *little- o function* is a monotonically increasing function $\delta : [0, \infty) \rightarrow [0, \infty)$ such that $\delta(t)/t \rightarrow 0$ as $t \rightarrow \infty$, and $|\delta(s) - \delta(t)| \leq |s - t|$ for all $s, t > 0$.

Definition 4.2. A *short function* is a decreasing function $D : [0, \infty) \rightarrow (0, 1]$, satisfying $D(t) \leq 1/t$ for $t > 1$, and $|D(s) - D(t)| \leq |s - t|$ for all $s, t > 0$.

The 1-Lipschitz condition forms part of both above definitions for technical reasons: in the case of Definition 4.1, it is used in the next section to force $\delta(\text{len}(\gamma))$ to be close to $\delta(d(o, x))$ when γ is a 1-short path from o to x , while in the case of Definition 4.2, it ensures that subsegments of a D -short segment are D -short (see Definition 4.3 and Lemma 4.4).

Note that the Lipschitz assumption in Definition 4.2 does not restrict the decay rate of a short function: if $E : [0, \infty) \rightarrow (0, 1]$ is any decreasing function such that $E(t) \leq 1/t$ for $t > 1$, and $D : [0, \infty) \rightarrow (0, 1]$ is the function which is affine on each interval $[n-1, n]$, $n \in \mathbb{N}$, and defined by the equation $D(n-1) = E(n)$, then D is a short function. Similarly the Lipschitz assumption in Definition 4.1 puts no restriction on how slowly or quickly δ increases, among the class of monotonically increasing functions satisfying $\delta(t)/t \rightarrow 0$, since we could define such a δ by piecewise linear interpolation of the values of a function f at 0 and $2^{n-1}A$, $n \in \mathbb{N}$, where f is any given non-negative function satisfying $f(t)/t \rightarrow 0$ as $t \rightarrow \infty$ and $A > 0$ is so large that $f(s)/s \leq 1/2$ for $t \geq A$.

Definition 4.3. Given a short function D , a segment from x to y is said to be *D-short* if it is h -short for $h = D(d(x, y))$.

Note that, although the two concepts of h -short and D -short segments create a potential ambiguity of terminology, the context will always indicate which sense of “short” we mean, and we also use the convention of using capital or lower-case letters to indicate whether we are talking about a short segment in this new sense or the old sense, respectively.

Lemma 4.4. *Every subsegment of a D-short segment is a D-short segment.*

Proof. Suppose $\gamma : [0, L] \rightarrow X$ is a D -short segment from x to y parametrized by arclength, and let $h := D(d(x, y))$, so that $L \leq d(x, y) + h$. Let $z_i := \gamma(t_i)$ for some $t_i \in [0, L]$, $i = 1, 2$, let λ be the associated subpath of γ , and let $M = |t_1 - t_2|$ be the length of λ . A subpath of a h -short segment is a h -short segment (just use the triangle inequality!), and so λ is a D -short segment if $d(z_1, z_2) \leq d(x, y)$.

If instead $\delta := d(z_1, z_2) - d(x, y) > 0$, then

$$M - d(z_1, z_2) \leq L - d(z_1, z_2) = L - d(x, y) - \delta \leq h - \delta$$

while $D(d(z_2, z_2)) \geq h - \delta$ by the Lipschitz property, and so λ is again D -short. \square

We are now ready to define our three variants of bouquets.

Definition 4.5. Suppose X is an $\text{rCAT}(0)$ length space. Let δ be a little- o function, D a short function, and $o \in X$. A *loose (δ, D) -bouquet from o* is a sequence β of unit speed D -short segments $\beta_n : [0, L_n] \rightarrow X$, $n \in \mathbb{N}$, with the following properties:

- (i) $\beta_n(0) = o$ for all $n \in \mathbb{N}$; we call o the *initial point* of β .
- (ii) (L_n) is monotonically increasing and has limit infinity.
- (iii) $d(\beta_m(t), \beta_n(t)) \leq \delta(t)$, for all $0 \leq t \leq L_m$, $m \leq n$, $m, n \in \mathbb{N}$.

We call the points $\beta_n(L_n)$ the *tips* of β .

Definition 4.6. A *(c, D) -bouquet from o* is a loose (δ, D) -bouquet from o for some constant function $\delta(t) \equiv c \geq 0$. A *standard bouquet from o* in a C - $\text{rCAT}(0)$ space means a $(2C + 2, D)$ -bouquet β from o , with $D(t) = 1/(1 \vee (2t))$, $t \geq 0$, and $L_n = (2C + 2)^n$, $n \in \mathbb{N}$.

Note that the definition of a standard bouquet depends on the C -parameter of the ambient $\text{rCAT}(0)$ space. In this definition, the precise choice $c = 2C + 2$ is a mere convenience, but choosing some fixed $c > 2C$ is important for the topological

arguments in Section 6. As for L_n , it is only important that we choose some sequence increasing to infinity but $L_n = (2C + 2)^n$ is technically convenient.

We often speak of (*loose/standard*) *bouquets*, dropping references to the initial point o and parameters c, δ, H , if these are unimportant. We denote by $\mathcal{B}(X)$, $\mathcal{LB}(X)$, $\mathcal{B}_{\text{std}}(X)$, the sets of all bouquets, loose bouquets, or standard bouquets (with basepoint o), respectively, so $\mathcal{B}_{\text{std}}(X) \subset \mathcal{B}(X) \subset \mathcal{LB}(X)$.

Definition 4.7. Let $\beta^i = (\beta_n^i)_{n=1}^\infty$, $i = 1, 2$, be a pair of loose bouquets in an rCAT(0) space X , where $\beta_n^i : [0, L_n^i] \rightarrow X$. Then β^1 and β^2 are said to be *loosely asymptotic*, denoted $\beta^1 \sim_L \beta^2$, if there is a little-o function δ such that

$$d(\beta_m^1(t), \beta_n^2(t)) \leq \delta(t), \quad 0 \leq t \leq L_m^1 \wedge L_n^2.$$

The equivalence class of loose bouquets loosely asymptotic to β will be denoted by $[\beta]_L$.

Definition 4.8. Bouquets β^1 and β^2 are said to be *asymptotic*, denoted $\beta^1 \sim \beta^2$, if they are loosely asymptotic for some constant little-o function $\delta(t) \equiv K \geq 0$. The equivalence class of bouquets asymptotic to β will be denoted by $[\beta]$.

Definition 4.9. Assuming X is an rCAT(0) space, we call $\partial_{\text{LB}}X := \mathcal{LB}(X)/\sim_L$ the *loose bouquet boundary* of X , and $\partial_{\text{B}}X := \mathcal{B}(X)/\sim$ the *bouquet boundary* of X .

Other variants of interest are $\mathcal{B}(X)/\sim_L$, $\mathcal{B}_{\text{std}}(X)/\sim_L$, and $\mathcal{B}_{\text{std}}(X)/\sim$. We will see that all five variants lead to naturally equivalent notions of a boundary at infinity (Corollary 4.19), that they are independent of the choice of basepoint o (Corollaries 4.17 and 4.19), and that they generalize the ideal boundary of a complete CAT(0) space (Theorem 4.20).

It is clear that \sim or \sim_L is an equivalence relation in each of the above five variants. Note that \sim is not an equivalence relation on $\mathcal{LB}(X)$ since the notion of asymptoticity must be at least as loose as the bound on $d(\beta_m(t), \beta_n(t))$ in order to have an equivalence relation; easy examples can be found in the Euclidean plane.

Let us pause to make a few remarks relating to the above definitions. First, note that if β^1, β^2 are loose (δ_1, D) -bouquets and $d(\beta_m^1(t), \beta_n^2(t)) \leq \delta(t)$ for one particular choice of m, n , then $d(\beta_m^1(t), \beta_n^2(t)) \leq \delta(t) + 2\delta_1(t)$ for all allowable choices of m, n . So if we do not care about the particular little-o function δ , we can write the loose asymptoticity condition as

$$d(\beta^1(t), \beta^2(t)) \leq \delta(t), \quad 0 \leq t < \infty,$$

where $\beta^i(t)$ can be interpreted as $\beta_n^i(t)$ for any single $n = n(t)$ for which $\beta_n^i(t)$ is defined. Using Lemma 3.6, this last inequality for fixed t implies that

$$d(\beta^1(s), \beta^2(s)) \leq \delta(t) + 2C, \quad 0 \leq s \leq t,$$

and so loose asymptoticity of β^1 and β^2 is equivalent to the formally weaker condition: there exists a little-o function δ such that

$$\liminf_{t \rightarrow \infty} \frac{d(\beta^1(t), \beta^2(t))}{\delta(t)} \leq 1.$$

It follows routinely from the triangle inequality and the fact that we are using 1-short segments that the bound $d(\beta_m(t), \beta_n(t)) \leq c$ in the definition of a bouquet is quantitatively equivalent to assuming the seemingly weaker condition $d_H(\beta_m, \beta_n|_{[0, L_m]}) \leq c'$,

where d_H indicates Hausdorff distance. In fact the latter condition for a given c' implies the former condition for $c = 2c' + 1$. Similarly the uniform bound on $d(\beta_m^1(t), \beta_n^2(t))$ in the definition of asymptotic bouquets is quantitatively equivalent to a uniform bound on the Hausdorff distance between β_m^1 and $\beta_n^2|_{[0, L_m^1]}$, assuming without loss of generality that $L_m^1 \leq L_n^2$. Similar comments apply to the definitions of loose bouquets and loose asymptoticity.

Definition 4.10. Suppose $\alpha := (\alpha_n)_{n=1}^\infty$ is a sequence of numbers, with $0 < \alpha_n \leq 1$, $n \in \mathbb{N}$, and suppose $\beta = (\beta_n)$ is a (loose) bouquet. The α -pruning of β is $\beta' = (\beta'_n)$, where $\beta'_n = \beta_n|_{[0, \alpha_n L_n]}$. If α is a constant sequence (a) , we may refer to the a -pruning of β in place of the α -pruning of β .

We now make three simple observations about ways to get (loose) bouquets (loosely) asymptotic to a given bouquet; we write “equivalent” in all cases instead of “(loosely) asymptotic”. The last of these three observations is the only one where we needed to use the $\text{rCAT}(0)$ condition, specifically in the form of Lemma 3.6.

Observation 4.11. Every subsequence of a (loose) bouquet β is a (loose) bouquet equivalent to β ; we call such a subsequence a *(loose) sub-bouquet*.

Observation 4.12. If β is a (loose) bouquet, then an α -pruning of β is also a (loose) bouquet as long as the sequence $(\alpha_n L_n)$ is increasing and has limit infinity. Whenever the α -pruning of β is a (loose) bouquet, it is equivalent to β . In particular if $0 < a < 1$, then the a -pruning of a (loose) bouquet β is always a (loose) bouquet equivalent to β .

Observation 4.13. If a (loose) bouquet β' from o' has the same sequence of tips as a (loose) bouquet β from o , then β and β' are (loosely) asymptotic.

4.B. Equivalence of bouquet boundary definitions.

Theorem 4.14. Let X be a C - $\text{rCAT}(0)$ space and let β be a (c, D) -bouquet from o in X . If $o' \in X$, $c' > C$, and D' is any short function, then there exists a (c', D') -bouquet from o' which is asymptotic to β .

Proof. Let $\beta = (\beta_n)_{n=1}^\infty$ with $\beta_n : [0, L_n] \rightarrow X$ as usual, and let $x_n := \beta_n(L_n)$. Let $y_n := \beta_n(M_n)$ where $0 \leq M_n \leq L_n$ is chosen so that $d(o, y_n) = d(o, x_n)/2$. By thinning out β if necessary, we assume that $L_1 \geq 1 + 4d(o, o')$, and that

$$L_{n+1} \geq L_n + 4d(o, o') + 3, \quad n \in \mathbb{N}.$$

It follows that

$$d(o, x_{n+1}) \geq d(o, x_n) + 4d(o, o') + 2, \quad n \in \mathbb{N}.$$

and so

$$\left. \begin{aligned} d(o, y_{n+1}) &\geq d(o, y_n) + 2d(o, o') + 1 \\ d(o', y_{n+1}) &\geq d(o', y_n) + 1 \end{aligned} \right\}, \quad n \in \mathbb{N}.$$

Also $d(o, x_1) \geq 4d(o, o')$, so $d(o, y_1) \geq 2d(o, o')$, and $d(o', y_1) \geq d(o, o')$.

We choose a collection of unit speed h'_n -short paths $\lambda_n : [0, M'_n] \rightarrow X$ from o' to y_n , where $h'_n := D'(d(o', y_n))/2$. Because

$$d(o', y_n) \geq d(o', y_1) \geq d(o, o'),$$

we see that

$$d(o, y_n) \leq d(o, o') + d(o', y_n) \leq 2d(o', y_n),$$

and so

$$h'_n \leq 1/2d(o', y_n) \leq 1/d(o, y_n).$$

Since also $h'_n \leq 1/d(o, o')$, we see that $h'_n \leq H(o, o', y_n)$, where H is as in (3.2). The shortness parameter for β_n is

$$D(d(o, x_n)) \leq 1/d(o, x_n) \leq 1/2d(o, y_n),$$

and we similarly see that $D(d(o, x_n)) \leq H(o, o', y_n)$. Because $d(o', y_{n+1}) \geq d(o', y_n) + 1$, we see that the sequence (M'_n) is monotonically increasing. Also $M'_n \rightarrow \infty$ simply because $L_n \rightarrow \infty$.

We now fix $n, m \in \mathbb{N}$ with $m \leq n$, and choose y_m^1 on $\beta_n|_{[0, M_n]}$, and y_m^2 on $\lambda_n|_{[0, M'_n]}$ so that $d(y_m^1, o) = d(y_m^2, o) = d(y_m, o)$. By Lemma 3.7, we see that

$$d(y_m^1, y_m^2) \leq C + d(o, o'),$$

and the (c, D) -bouquet condition ensures that $d(y_m^1, y_m) \leq 1 + c$. Thus

$$(4.15) \quad d(y_m, y_m^2) \leq 1 + c + C + d(o, o').$$

Using the C -rCAT(0) condition we readily deduce that $\lambda := (\lambda_n)$ is a (c'', D') -bouquet from o' that is asymptotic to β , where $c'' = 1 + c + 2C + d(o, o')$. If c'' is larger than c' , we simply replace λ by the a -pruning of λ , where $a = (c' - C)/(1 + c + C + d(o, o'))$. \square

Remark 4.16. The above proof works just as well if (y_n) is any other sequence of points such that y_n lies on β_n , $d(o, y_n) \leq d(o, x_n)/2$, and $(d(o, y_n))$ is unbounded, although we might need to select a subsequence to ensure that $d(o, y_{n+1})$ and $d(o', y_{n+1})$ increase quickly enough. Alternatively if $D(t) \leq 1/(1 \vee 2t)$, we could use paths to x_n instead of paths to y_n , since paths to y_n were needed only to ensure that the rCAT(0) condition could be applied to the resulting triangle. The latter variant will prove useful in Section 6.

Corollary 4.17. *If X is an rCAT(0) space, then $\mathcal{B}_{std}(X)/\sim$ can be identified with $\partial_B X = \mathcal{B}(X)/\sim$, and both are independent of the basepoint o .*

Proof. Let β be a bouquet from o in X . By Theorem 4.14, there exists a $(2C + 2, D)$ -bouquet β' from any other point $o' \in X$ that is asymptotic to β , where $D(t) = 1/(1 \vee (2t))$, $t \geq 0$. If the associated lengths L'_n of β'_n are not as required, taking a subsequence allows us to assume that they are at least as large as required, and then we get a standard bouquet by suitably pruning this bouquet. The result now follows easily. \square

We next prove the equivalence of the bouquet boundary and the loose bouquet boundary.

Theorem 4.18. *If X is an rCAT(0) space, then the identity map from $\mathcal{B}(X)$ to $\mathcal{LB}(X)$ induces a natural bijection $i_L : \partial_B X \rightarrow \partial_{LB} X$.*

Proof. Suppose X is C -rCAT(0). Trivially $i_L([\beta]) := [\beta]_L$ is well-defined. We next prove that i_L is injective. Suppose that $i_L([\beta^1]) = i_L([\beta^2])$ for a pair of (c, D) -bouquets β^1, β^2 . By Theorem 4.14, we may assume that β^1, β^2 have a common initial point o . Then there exists a little-o function δ such that

$$d(\beta_{m_t}^1(t), \beta_{n_t}^2(t)) \leq \delta(t),$$

where the indices m_t, n_t are such that $L_{m_t}^1 \wedge L_{n_t}^2 \geq t$, where $L_m^i = \text{len}(\beta_m^i)$ as usual. Using Lemma 3.6, we deduce that for all indices $m, n \in \mathbb{N}$,

$$d(\beta_m^1(s), \beta_n^2(s)) \leq s \frac{\delta(t)}{t} + 2C + 2c, \quad 0 \leq s \leq L_m^1 \wedge L_n^2 \wedge t.$$

Letting t tend to infinity, we deduce that

$$d(\beta_m^1(s), \beta_n^2(s)) \leq 2C + 2c, \quad 0 \leq s \leq L_m^1 \wedge L_n^2,$$

and so $[\beta^1] = [\beta^2]$, as required.

Finally, we prove that i_L is surjective. Suppose $\beta = (\beta_n)$ is a loose (δ, D) -bouquet, with $\beta_n : [0, L_n] \rightarrow X$, $n \in \mathbb{N}$ as usual. We choose a strictly increasing sequence of positive integers $(n_k)_{k=1}^\infty$ such that $L_{n_k} \geq k$ and $k\delta(L_{n_k}) \leq L_{n_k}$. We let $\beta' = (\beta'_k)_{k=1}^\infty$ be the α -pruning of the sub-bouquet (β_{n_k}) , where $\alpha = (\alpha_k)_{k=1}^\infty$ is defined by $\alpha_k = k/L_{n_k}$. Then β'_k has length k . Using Lemma 3.6, we deduce that for all $m \leq n$,

$$d(\beta'_m(s), \beta'_n(s)) \leq C + \alpha_m \delta(L_{n_m}) \leq C + 1, \quad 0 \leq s \leq m,$$

Thus β' is a $(1 + C, D)$ -bouquet and, since it is a pruning of a sub-bouquet of β , it is loosely asymptotic to β . Thus $i_L([\beta']) = [\beta]_L$, as required. \square

Corollary 4.19. *There are natural identification maps between the boundary variants $\mathcal{LB}(X)/\sim_L$, $\mathcal{B}(X)/\sim_L$, $\mathcal{B}_{\text{std}}(X)/\sim_L$, $\mathcal{B}(X)/\sim$, and $\mathcal{B}_{\text{std}}(X)/\sim$ of an $\text{rCAT}(0)$ space X .*

Proof. Let $q_1 : \mathcal{B}(X) \rightarrow \partial_B X = \mathcal{B}(X)/\sim$ and $q_2 : \mathcal{LB}(X) \rightarrow \partial_{LB} X = \mathcal{LB}(X)/\sim_L$ be the defining quotient maps. By Corollary 4.17, $q_3 := q_1|_{\mathcal{B}_{\text{std}}(X)} : \mathcal{B}_{\text{std}}(X) \rightarrow \partial_B X$ is surjective. By Theorem 4.18, the identity map $i : \mathcal{B}(X) \hookrightarrow \mathcal{LB}(X)$ induces a natural identification $i_L : \partial_B X \rightarrow \partial_{LB} X$, and so the following diagram commutes.

$$\begin{array}{ccccc} \mathcal{B}_{\text{std}}(X) & \hookrightarrow & \mathcal{B}(X) & \xhookrightarrow{i} & \mathcal{LB}(X) \\ & \searrow q_3 & \downarrow q_1 & & \downarrow q_2 \\ & & \partial_B X & \xhookrightarrow{i_L} & \partial_{LB} X \end{array}$$

Each of the five types of boundary is either the image of q_i , $1 \leq i \leq 3$, or the image of $i_L \circ q_i$, $i = 1, 3$, and they can all be identified because the maps q_i are all surjective and i_L is bijective. \square

4.C. The ideal boundary versus the bouquet boundary. In an $\text{rCAT}(0)$ space X , a geodesic ray $\gamma : [0, \infty) \rightarrow X$ can be identified with the bouquet $(\gamma_n)_{n=1}^\infty$, where γ_n is the initial segment of length n of γ , parametrized by arclength. This gives rise to a natural injection $i_I : \partial_I X \rightarrow \partial_B X$.

Theorem 4.20. *If X is a complete $\text{CAT}(0)$ space, then the natural injection $i_I : \partial_I X \rightarrow \partial_B X$ is bijective.*

Before proving Theorem 4.20, we give simple examples to show that the ideal boundary is not as well behaved in $\text{rCAT}(0)$ spaces, or even in incomplete $\text{CAT}(0)$ spaces, as it is in complete $\text{CAT}(0)$ spaces. Such pathologies would be known to experts. The point of giving them here is to contrast them with Theorem 4.14 which says that no such pathologies arise with the bouquet boundary of an $\text{rCAT}(0)$ space.

Example 4.21. Let X be the metric subspace of the Euclidean plane given as follows using Cartesian coordinates:

$$X = \{(0, 0)\} \cup (0, 1) \times (0, \infty).$$

Then, as a convex subset of the Euclidean plane, X is CAT(0). It is also clear that $\partial_B X$ is a singleton set, as is $\partial_I X$ if we define it as the set of equivalence classes of geodesic rays. There is however no geodesic ray from $o = (0, 0)$.

Example 4.22. Let X_i be an isometric copy of the space X in Example 4.21, and let Y_I be the metric space obtained as a quotient of the disjoint union $\bigcup_{i \in I} X_i$ where we identify every copy of $(0, 0)$, and I is some nonempty index set. Then Y_I is CAT(0) and there is a natural bijection from $\partial_B Y_I$ to I ; the same can be said of $\partial_I X$ if we define it as the set of equivalence classes of geodesic rays. However, there is only one geodesic ray from each $y \in Y_I$, $y \neq o$, and none at all from o .

In the previous pair of examples, we could identify the ideal and bouquet boundaries, even if the ideal boundary was not as well behaved. The next example shows that the situation can be worse than this.

Example 4.23. Let X be the subset of the Euclidean plane given as follows

$$X = \left(\bigcup_{i=0}^{\infty} \{(i, 0)\} \right) \cup \left(\bigcup_{i=1}^{\infty} (i-1, i) \times (0, 1) \right).$$

Then we claim that X is CAT(0). To see this, suppose that x, y, z are fixed but arbitrary points in X . If these points can be connected by a geodesic triangle, then any such triangle must clearly be contained in some “initial part” of X having the form $X_n = \bigcup_{i=1}^n A_i$, where

$$A_i = \{(i-1, 0), (i, 0)\} \cup (i-1, i) \times (0, 1), \quad 1 \leq i \leq n.$$

Since each A_i is a convex subset of the plane, it is CAT(0) in the induced metric. Since X_n is obtained by a finite succession of isometric gluings of the sets A_i along closed convex subsets (in fact along singleton sets!), it follows from the basic gluing theorem II.11.1 of [1] that X_n is CAT(0). In particular there exists at least one geodesic triangle with vertices x, y, z , and all such triangles satisfy the CAT(0) inequality. Since x, y, z are arbitrary, we deduce that X is CAT(0). It is also clear that the bouquet boundary is a singleton set, but that X contains no geodesic ray. By joining isometric copies of X at $o = (0, 0)$, we can get a space whose bouquet boundary has any desired cardinality, but whose ideal boundary is empty.

Theorem 4.20 is an immediate consequence of the following generalization to bouquets of a well-known result concerning geodesic rays; the proof is a modification of that of Theorem 4.14.

Theorem 4.24. *Let X be a complete CAT(0) space and let β be a (c, D) -bouquet from o in X . Given any $o' \in X$, there exists a geodesic ray parametrized by arclength $\gamma : [0, \infty) \rightarrow X$ with $\gamma(0) = o'$ and which is asymptotic to β in the sense that there exists a constant c' such that*

$$d(\gamma(t), \beta_n(t)) \leq c', \quad 0 \leq t \leq L_n, \quad n \in \mathbb{N},$$

where as usual $L_n = \text{len}(\beta_n)$.

Proof. By Theorem 4.14, there exists a (c_2, D') -bouquet $\beta' = (\beta'_n)$ from o' that is asymptotic to β , where c_2 is as in the proof of that result and D' is an arbitrary short function to be fixed below. If necessary, we take a subsequence of β' to ensure that the associated sequence of path lengths (L'_n) are such that $L'_{n+1} \geq 4^n L'_n > 0$. For each $n \in \mathbb{N}$, we then prune β'_n by a factor $\alpha_n := 2^{-n}$ to get a path β''_n of length L''_n which increases to infinity. For $n_0 \in \mathbb{N}$, the sequence $(\beta''_{n+n_0})_{n=0}^\infty$ is a $(2^{-n_0+1}c_2, D')$ -bouquet: this follows from Lemma 3.6 with $a_1 = a_2 = o'$. Note that the parameter $2^{-n_0+1}c_2$ is twice as large as would be needed for $C' = 0$ in order to incorporate the C' term; for this to suffice, we of course need that $C' > 0$ be sufficiently small, but this can be guaranteed by choosing a sufficiently small short function D' above.

It follows that the sequence $\beta'' = (\beta''_n)$ converges in the pointed Hausdorff sense to a path γ . Since β''_n is h_n -short where $h_n = D'(d(o, y_n)) \rightarrow 0$ as $n \rightarrow \infty$, it follows that γ is a geodesic ray.

Since β' , and so also β'' , is asymptotic to β , it is clear that γ is asymptotic to β . \square

4.D. The end boundary and the bouquet boundary. By an *end* of a metric space X (with basepoint o), we mean a sequence (U_n) of components of $X \setminus \bar{B}_n$, where $\bar{B}_n = \overline{B(o, n)}$ for fixed $o \in X$ and $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. We do not require \bar{B}_n to be compact. We denote by $\partial_E X$ the collection of ends of X and call it the *end boundary* of X .

Ends with respect to different basepoints are compatible under set inclusion: defining U_n, V_n for all $n \in \mathbb{N}$ to be components of $X \setminus \overline{B(o, n)}$ and $X \setminus \overline{B(o', n)}$, respectively, it is clear that U_n is a subset of a unique V_m whenever $n - m > d(o, o')$. This compatibility gives rise to a natural bijection between ends with respect to different basepoints, allowing us to identify them and treat the end boundary as being independent of the basepoint.

A *finite ϵ -net* for a subset A of a metric space X is a set $S \subset X$ of finite cardinality such that every $x \in A$ lies in the ball $B(x, \epsilon)$ for some $x \in S$. Requiring A to have a finite ϵ -net for fixed $\epsilon > 0$ is strictly weaker than requiring A to be totally bounded.

We now examine the relationship between the end boundary and the bouquet boundary of an $\text{rCAT}(0)$ space.

Theorem 4.25. *If X is an $\text{rCAT}(0)$ space, then there is a natural map $\eta : \partial_B X \rightarrow \partial_E X$. If additionally there exists $\epsilon > 0$ such that every ball in X has a finite ϵ -net, then*

- (a) η is surjective;
- (b) $\partial_B X$ and $\partial_E X$ are nonempty if and only if X is unbounded.

The assumption that balls have finite ϵ -nets is essentially stating that balls are “totally bounded at a fixed scale ϵ ”. This holds in particular if X is proper, so the above result implies Theorem 1.3. Without this assumption both conclusions (a) and (b) in Theorem 4.25 can fail. In the case of (b) this is easy to see: the union of segments from the origin to $(n, 1)$, $n \in \mathbb{N}$, in the Euclidean plane is unbounded but clearly both $\partial_B X$ and $\partial_E X$ are empty. For the failure of (a), even for complete $\text{CAT}(-1)$ spaces (a condition which implies both $\text{CAT}(0)$ and Gromov hyperbolic), we refer the reader to [4, Theorem 2].

Proof of Theorem 4.25. Let \mathcal{U}_n denote the set of components of $X \setminus \overline{B(o, n)}$ for fixed $o \in X$, and let $\beta = (\beta_n)$ be a bouquet with initial point o , where $\beta_n : [0, L_n] \rightarrow X$ as

usual. We claim that for each $m \in \mathbb{N}$ there exists $t_0 = t_0(m) > 0$ and $U_m \in \mathcal{U}_m$ such that $\beta_n(t) \in U_m$ whenever $n \in \mathbb{N}$ and $t_0 < t \leq L_n$. In fact if β is a (c, D) -bouquet from o , then we can take $t_0 = m + 3/2 + c/2$.

To justify the claim, suppose first that $x := \beta_n(t) \in U_m$ and $x' := \beta_n(t') \in U'_m$, where U_m, U'_m are distinct elements of \mathcal{U}_m , and $t' > t > t_0 = m + 3/2 + c/2$. Any path from x to x' must pass through $\overline{B(o, m)}$, and β_n is 1-short, so

$$d(x, x') \geq t - 1 - m + t' - 1 - m = t' - t + 2(t - m - 1) > t' - t + 1,$$

contradicting the fact that the segment $\beta|_{[t, t']}$ is 1-short.

Suppose next that $x := \beta_n(t) \in U_m$ and $x' := \beta_{n'}(t) \in U'_m$, where U_m, U'_m are distinct elements of \mathcal{U}_m , and $t' > t > t_0 = m + 3/2 + c/2$. As before, $d(x, x') \geq 2(t - m - 1) > 1 + c$, contradicting part (iii) of the definition of a (c, D) -bouquet. Putting together these last two proofs by contradiction we deduce the claim.

Fixing $\beta \in \mathcal{B}(X)$, it similarly follows that if $\beta' \in \mathcal{B}(X)$ is equivalent to β , and $\beta'_n : [0, L'_n] \rightarrow X$ as usual, then for each $m \in \mathbb{N}$ there exists $t_0 = t_0(m, \beta') > 0$ and $U_m \in \mathcal{U}_m$ such that $\beta'_n(t) \in U_m$ whenever $n \in \mathbb{N}$ and $t_0 < t \leq L'_n$. In fact if K is the constant from Definition 4.8, then we can take $t_0 = m + 3/2 + K/2$.

Thus we get a well-defined natural map $\eta : \partial_B X \rightarrow \partial_E X$ by taking $\eta(x) = (U_n)$ for $x = [(\beta_n)] \in \partial_B X$, where $U_m \in \mathcal{U}_m$ for $m \in \mathbb{N}$ is defined by the requirement that $\beta_n(t) \in U_m$ whenever t is sufficiently large, and n is so large that $t \leq L_n$.

Fixing $\epsilon > 0$, we now assume that every ball in X has a finite ϵ -net, and that C is the rCAT(0) constant of X . We also fix a basepoint $o \in X$. We claim that if (x_n) is any sequence in X such that $d(o, x_n)$ is an increasing function of n and $d(o, x_n) \geq n$, then there exists an $(\epsilon + C, D)$ -bouquet $\gamma = (\gamma_n)$ such that the tips of γ are located on 1-short paths from o to some subsequence of (x_n) .

Let D be a short function, and let us pick a sequence of D -short unit speed paths (β_n) from o to x_n . Writing $\beta_{0,n} := \beta_n$, $n \in \mathbb{N}$, the existence of a finite ϵ -net for the ball $\{x \in X \mid d(o, x) \leq n\}$ allows us to pick a subsequence $(\beta_{k,n})_{n=1}^\infty$ of $(\beta_{k-1,n})_{n=1}^\infty$ inductively so that the points $(\beta_{k,n}(k))$ all lie in a ball of radius ϵ . We now take $\gamma_n : [0, n] \rightarrow X$ to be the initial segment of $\beta_{n,n}$ for each $n \in \mathbb{N}$. It follows readily from Lemma 3.6 that $(\gamma_n)_{n=1}^\infty$ is an $(\epsilon + C, D)$ -bouquet, and so the claim follows.

We now apply the above procedure to each end (U_n) : just restrict the initial sequence (x_n) so that $x_n \in U_n$. Then $\gamma_n(n)$ is in the same component of $X \setminus \overline{B(o, n-1)}$ as $x_{n,n}$, and it then readily follows that $\eta([\gamma]) = (U_n)$. We have therefore proved (a).

Suppose X is unbounded, so there exists a sequence (x_n) in X such that $d(o, x_n)$ is an increasing function of n and $d(o, x_n) \geq n$. By the claim it follows that $\partial_B X$ is nonempty, and so $\eta(\partial_B X) \subset \partial_E X$ is also nonempty. Conversely if X is bounded, it is trivial that $\partial_B X$ and $\partial_E X$ are empty. This concludes the proof of (b). \square

Remark 4.26. Using the concept of loose asymptoticity, the above proof can be adapted to give the same conclusions if the ϵ -net assumption is replaced by the following weaker one: for a sequence of values $r_n \rightarrow \infty$, the balls $B(o, r_n)$ can be covered by a finite collection of balls of radius $\delta(r_n)$, where δ is a little- o function. An example of a space satisfying this assumption for every such sequence (r_n) is the following subspace X of l^2 :

$$X = \{x \in l^2 \mid \|q_1(x)\|_2 \leq \delta(|p_1(x)|)\}$$

where p_1 is projection onto the first coordinate, $q_1 = I - p_1$ is the complementary projection, and δ is a little-o function.

Remark 4.27. Although Theorem 4.25 tells us that η is surjective for many nice spaces, η quite often fails to be injective in such cases. For instance $\partial_I \mathbb{R}^2 = \partial_B \mathbb{R}^2$ can be identified with the unit circle, but the Euclidean plane \mathbb{R}^2 has only one end. To deduce that $\eta : \partial_B X \rightarrow \partial_E X$ is injective, it would suffice to assume some sort of mild bottleneck condition in each end. In fact if (U_n) is an end in an $\text{rCAT}(0)$ space X , and if there exists:

- a strictly increasing sequence of integers (n_k) ,
- a sequence of points (x_k) with $x_k \in U_{n_k}$,
- a little-o function δ , and
- a sequence of positive numbers (r_k) such that $r_k \leq \delta(n_k)$ and such that every path from o to $U_{n_k} \setminus B(x_k, r_k)$ must pass through $B(x_k, r_k)$,

then there is a unique $x \in \partial_B X$ such that $\eta(x) = (U_n)$. For existence, we show that a sequence of suitably short paths from o to x_n , $n \in \mathbb{N}$, gives a loose bouquet, and for uniqueness, we use Lemma 3.6. We leave the details to the reader.

Remark 4.28. However a bottleneck condition of the type considered in Remark 4.27 is not necessary for η to be injective. To see this, we first define

$$X_0 := \{(x, y) \in \mathbb{R}^2 : |y| \leq x \leq 1\},$$

$$X_i := \{(x, y) \in \mathbb{R}^2 : |y| \leq x, 2^{i-1} \leq x \leq 2^i\}, \quad i \in \mathbb{N}.$$

We attach the Euclidean metric to each of the above sets. For all $i \geq 0$, X_i is a convex subset of the Euclidean plane, and so it is $\text{CAT}(0)$. Now let X be the space obtained by isometrically gluing X_{i-1} to X_i , $i \in \mathbb{N}$, according to the following rule: if i is odd, we glue along the line segment of points $(2^{i-1}, y) \in \mathbb{R}^2$, $0 \leq y \leq 2^{i-1}$, while if i is even, we glue along the line segment of points $(2^{i-1}, y) \in \mathbb{R}^2$, $-2^{i-1} \leq y \leq 0$. Each gluing is along an isometric pair of closed convex subsets of the complete spaces X_{i-1} and X_i , and it follows as in Example 4.23 that X is a complete $\text{CAT}(0)$ space. It is also clear that X does not satisfy a bottleneck condition, that $\partial_I X = \partial_B X$ is a singleton set, and so $\eta : \partial_B X \rightarrow \partial_E X$ is injective.

5. SEQUENTIAL CONSTRUCTIONS

In this section we show that in a Gromov hyperbolic space, the bouquet boundary can be naturally identified with the Gromov boundary. Since the Gromov boundary is defined using sequences that “march off to infinity”, we embed this proof in a wider discussion of ways to define the bouquet boundary in a general $\text{rCAT}(0)$ space using such sequences.

As in the previous section, we can restrict these sequences in either a tight or loose manner, depending on whether certain quantities are bounded or grow more slowly than distance to the origin, and for the tight case we can use a tight or loose equivalence. In this way we get three notions of boundary at infinity. We will see that only two of them can be naturally identified with the bouquet boundary, although all three of them can be naturally identified with both the bouquet and the Gromov boundary in the case of a Gromov hyperbolic space. As in the previous section, we

are only talking about set theoretic identifications: we discuss an associated topology in Section 6.

We first record two lemmas. The first can be proved in the same manner as its short arc variant in [13, 2.33].

Lemma 5.1. *Suppose that λ is a h -short path from x to y in a metric space X , and that $z \in X$. Then $\langle x, y; z \rangle \leq \text{dist}(z, \lambda) + h/2$. If additionally X is δ -hyperbolic, then $\text{dist}(z, \lambda) \leq \langle x, y; z \rangle + h + 2\delta$.*

The second lemma that we need is the so-called *Tripod Lemma* for hyperbolic spaces. This version is as stated in [13, 2.15], except that again we are using short paths rather than short arcs.

Lemma 5.2. *Suppose that γ_1 and γ_2 are unit speed h -short paths from o to x_1 and x_2 , respectively, in a δ -hyperbolic space. Let $u_1 = \gamma_1(t)$ and $u_2 = \gamma_2(t)$ for some $t \geq 0$, where $d(o, u_1) \leq \langle x_1, x_2; o \rangle$. Then $d(u_1, u_2) \leq 4\delta + 2h$.*

As defined in Definition 2.2, the notion of a sequence that “marches off to infinity” is played by a Gromov sequence, defined as a sequence (x_n) such that $\langle x_m, x_n; o \rangle \rightarrow \infty$ as $m, n \rightarrow \infty$, and the Gromov boundary $\partial_G X$ is defined by an associated equivalence relation. This definition is not however consistent with the bouquet boundary; see Proposition 5.19 below. Instead we proceed as follows.

If β is a loose (δ, D) -bouquet with initial point o in an $\text{rCAT}(0)$ space X , and $x = f(\beta)$ is the sequence of tips of β (so $x = (\beta_n(L_n))$), then it follows from Lemma 5.1 that

$$\langle o, x_n; x_m \rangle \leq \delta'(d(o, x_m)), \quad \text{for all } m, n \in \mathbb{N}, m \leq n.$$

where $\delta'(t) = \delta(t) + 1 + 1/2$. Notice that the term 1 in $\delta'(t)$ bounds the difference $|\delta(L_m) - \delta(d(o, x_m))| \leq |L_m - d(o, x_m)|$, where we use the Lipschitz property of δ ; we make similar estimates in future without comment. The term $1/2$ is an upper bound for $D(d(o, x_n))$; it could of course be replaced by $D(d(o, x_1))/2 \leq 1/2$.

Definition 5.3. Given a little- o function δ , a *loose δ -bouquet sequence* (with basepoint o) is a sequence $(x_n)_{n=1}^\infty$ in an $\text{rCAT}(0)$ space X such that $(d(o, x_n))_{n=1}^\infty$ is an unbounded monotonically increasing sequence, and such that

$$\langle o, x_n; x_m \rangle \leq \delta(d(o, x_m)), \quad \text{for all } m, n \in \mathbb{N}, m \leq n.$$

As before, we define a *c-bouquet sequence* (with basepoint o) as above but with δ equal to some constant function $c \geq 0$. We denote by $\mathcal{LBS}(X)$ and $\mathcal{BS}(X)$ the sets of loose bouquet sequences and bouquet sequences, respectively, in both cases with basepoint o ; we omit the basepoint from this notation except in Observation 5.7 where we note that these notions are essentially independent of the basepoint.

By the discussion before the above definition, we see that if β is a loose (δ, D) -bouquet with initial point o in an $\text{rCAT}(0)$ space X , and $x = f(\beta)$ is the sequence of tips of β (so $x = (\beta_n(L_n))$), then x is a loose δ' -bouquet sequence with basepoint o , with δ' as above. We call $f : \mathcal{LB}(X) \rightarrow \mathcal{LBS}(X)$ the *tip map* of X (at basepoint o); f also maps $\mathcal{B}(X)$ to $\mathcal{BS}(X)$.

We now define the associated notions of boundary at infinity in the natural way. Although these notions are defined in terms of a basepoint o , they will turn out to be independent of o ; see Observation 5.7.

Definition 5.4. Fixing a basepoint $o \in X$, we define the *loose bouquet sequence boundary of X* , $\partial_{\text{LBS}}X$, to be the set of equivalence classes given by all loosely asymptotic bouquet sequences with basepoint o , where two loose bouquet sequences $x = (x_n)$ and $y = (y_n)$ are *loosely asymptotic*, written $x \sim_{\text{LS}} y$, if there exists a little-o function δ such that

$$\langle o, x_m; y_n \rangle \wedge \langle o, y_n; x_m \rangle \leq \delta(d(o, x_m) \wedge d(o, y_n)), \quad m, n \in \mathbb{N},$$

We denote by $[x]_{\text{LS}}$ the equivalence class in $\partial_{\text{LBS}}X$ containing a given loose bouquet sequence x .

Definition 5.5. Similarly, we define the *bouquet sequence boundary of X* , $\partial_{\text{BS}}X$, to be the set of equivalence classes given by all asymptotic bouquet sequences with basepoint o , where two bouquet sequences $x = (x_n)$ and $y = (y_n)$ are *asymptotic*, written $x \sim_{\text{S}} y$, if they are loosely asymptotic for δ equal to some constant function $K \geq 0$. We denote by $[x]_{\text{S}}$ the equivalence class in $\partial_{\text{BS}}X$ containing a given bouquet sequence x .

We now make a couple of observations about ways of constructing new (loose) bouquet sequences that are equivalent to a given bouquet sequence.

Observation 5.6. Any subsequence of a (loose) bouquet sequence x is a (loose) bouquet sequence that is (loosely) asymptotic to x ; we refer to such a subsequence as a *(loose) bouquet subsequence*.

Observation 5.7. A c -bouquet sequence $x = (x_n)$ with basepoint o may not be a bouquet sequence for another basepoint o' , since $d(o', x_n)$ might not be increasing. However, by thinning out x to get a subsequence $y = (y_n)$ where $d(o, y_{n+1}) \geq d(o, y_n) + 2d(o, o')$, it follows easily from the triangle inequality that y is a c' -bouquet sequence with basepoint o' , where $c' = c + d(o, o')$, and that $y \sim_{\text{S}} x$. In a similar fashion, by taking a suitable subsequence of $x \in \mathcal{LBS}_o(X)$, we get a sequence $y \in \mathcal{LBS}_{o'}(X)$, with $y \sim_{\text{LS}} x$. Thus $\partial_{\text{BS}}X$ and $\partial_{\text{LBS}}X$ are independent of the basepoint o .

Theorem 5.8. Suppose X is an $r\text{CAT}(0)$ space. Then the tip map $f : \mathcal{LB}(X) \rightarrow \mathcal{LBS}(X)$ induces a natural bijection $i_{\text{S}} : \partial_{\text{LB}}X \rightarrow \partial_{\text{LBS}}X$. Consequently, $\partial_{\text{LBS}}X$ is naturally bijective with $\partial_{\text{B}}X$, and also with $\partial_{\text{I}}X$ if X is $\text{CAT}(0)$.

Proof. The first statement of our theorem follows from the claim that $i_{\text{S}}([\beta]_{\text{L}}) = [x]_{\text{LS}}$ is a well-defined map, where $\beta = (\beta_n) \in \mathcal{LB}(X)$ and $x = (x_n) = f(\beta)$. Because of Observations 4.11 and 5.6, it suffices to prove the claim after taking any desired subsequence, so we may assume that the length of β_n grows as fast as desired.

To establish the claim, we suppose $\beta^1 = (\beta_n^1)$ and $\beta^2 = (\beta_n^2)$ are loosely asymptotic loose (δ, D) -bouquets, with δ_1 being the little-o function in Definition 4.7 for this pair of loose bouquets, and we denote the length of β_n^i by L_n^i , $i = 1, 2$. Let $x = (x_n) = f(\beta^1)$ and $y = (y_n) = f(\beta^2)$. Let o and o' be the initial points of β^1 and β^2 , respectively. We know that x and y are loose bouquet sequences with basepoints o and o' , respectively, and so both are loose bouquet sequences from o once we suitably thin out β^2 (and so y), as we may.

Fixing arbitrary $m, n \in \mathbb{N}$, assume first that $d(o, x_m) \leq d(o, y_n)$. Now

$$\text{dist}(x_m, \beta_n^2) \leq \text{dist}(x_m, \beta_n^2(L_m^1)) \leq \delta(L_m^1).$$

By Lemma 5.1, we deduce that $\langle o', y_n; x_m \rangle \leq \delta(L_m^1) + 1/2$, and so

$$\langle o, y_n; x_m \rangle \leq \delta(L_m^1) + d(o, o') + 1/2 \leq \delta(d(o, x_m)) + d(o, o') + 3/2.$$

If instead $d(o, x_m) > d(o, y_n)$, we similarly get $\langle o, x_m; y_n \rangle \leq \delta(d(o, y_n)) + 3/2$. Since m, n are arbitrary, it follows that x and y are loosely asymptotic and the claim is established.

We next show that i_S is injective. Suppose $i_S([\beta^1]_L) = i_S([\beta^2]_L)$, where $x = (x_n)$, $y = (y_n)$, and (L_n^i) , $i = 1, 2$, are as before, except now instead of assuming that β^1 and β^2 are loosely asymptotic, we want to prove this. By taking subsequences if necessary, we assume that

$$L_{n+1}^1 > L_n^2 + 2 + \delta(L_n^2) > L_n^1 + 4 + \delta(L_n^2) + \delta(L_n^1), \quad n \in \mathbb{N},$$

where δ is the loose asymptoticity parameter for x, y in Definition 5.4. By the triangle inequality, this last pair of inequalities ensures that $\langle o, x_m; y_n \rangle > \delta(L_m^1)$ whenever $n \geq m$ and $\langle o, y_n; x_m \rangle > \delta(L_n^2)$ whenever $m > n$, so loose asymptoticity of x and y tells us that $\langle o, y_n; x_m \rangle \leq \delta(L_m^1)$ whenever $n \geq m$ and $\langle o, x_m; y_n \rangle \leq \delta(L_n^2)$ otherwise.

To show that β^1 and β^2 are loosely asymptotic, we bound $d(\beta_m^1(t), \beta_n^2(t))$ by a suitably little-o quantity for $m \leq n$ and $t \leq L_m^1$; the corresponding result when $m > n$ is handled similarly. It suffices to establish such a bound when $t = L_m^1$ (and so $\beta_m^1 = x_m$), since a corresponding bound for smaller t follows from Lemma 3.6. Without loss of generality, we assume that $d(o, x_m) \geq 1$ and $d(o, y_n) \geq 1$.

Since $\langle o, y_n; x_m \rangle \leq \delta(L_m^1)$, the concatenation of β_m^1 and a D -short path γ from x_m to y_n gives a K -short path from o to y_n , where $K := 2\delta(L_m^1) + 2$. Inequality (3.5) says that

$$(d(x_m, u) - C)^2 \leq (1 - t)(d(x_m, o))^2 + t(d(x_m, y_n))^2 - t(1 - t)(d(o, y_n))^2,$$

where $u := \beta_n^2(s)$, $s := L_m^1$, and $0 \leq t \leq 1$ is any number satisfying $td(o, y_n) \leq s$ and $(1 - t)d(o, y_n) \leq L - s$. This last pair of inequalities holds for $t = s/L$, where $L := L_n^2$. Writing $\Delta := d(o, y_n)$, we note that $d(x_m, o) \leq s$ and $d(x_m, y_n) \leq \text{len}(\gamma) \leq d(o, y) + K - s \leq L + K - s$, so

$$(5.9) \quad (d(x_m, u) - C)^2 \leq s^2(1 - s/L) + (s/L)(L - s + K)^2 - t(1 - t)\Delta^2.$$

Contrasting the inequalities $t\Delta \leq s$ and $(1 - t)\Delta \leq L - s$ with $\Delta \geq L - 1/\Delta$, we see that we can almost reverse the first two inequalities: $t\Delta \geq s - 1/\Delta$ and $(1 - t)\Delta \geq L - s - 1/\Delta$. Thus

$$t(1 - t)\Delta^2 \geq (s - 1/\Delta)(L - s - 1/\Delta) \geq s(L - s) - 2.$$

Since also $s^2(1 - s/L) + (s/L)(L - s)^2 = s(L - s)$, it follows from (5.9) that

$$(d(x_m, u) - C)^2 \leq 2 + s(2(L - s)K + K^2)/L \leq 2 + 2sK + K^2.$$

From this it readily follows that $d(x_m, u)$ is a little-o function of s , as required.

Next we prove that i_S is surjective. Fix a loose bouquet sequence $x = (x_n)$. By taking a subsequence if necessary, we may assume that $d(o, x_{n+1}) \geq d(o, x_n) + 1$ for all n . We now construct unit speed D -short paths β_n from o to x_n for each n and claim that the resulting sequence $\beta = (\beta_n)$ is a loose bouquet such that $i_S([\beta]_L) = [x]_L$. The only non-trivial part of the proof is that $d(\beta_m(t), \beta_n(t))$ is dominated by some little-o function of t . The proof of this goes along the same lines as the proof of injectivity, so we leave it to the reader.

The final statement in the theorem follows from Corollary 4.19 and Theorem 4.20. \square

Remark 5.10. A fact that will be useful in the next section is that the restricted tip map $g := f|_{\mathcal{B}_{\text{std}}(X)}$ induces the bijection $i_S : \partial_{\text{LB}}X \rightarrow \partial_{\text{LBS}}X$. This follows from Corollary 4.19 and the proof of Theorem 5.8. Thus the following diagram commutes, where surjectivity is as usual denoted by a double arrow.

$$\begin{array}{ccccc} \mathcal{B}_{\text{std}}(X) & \xrightarrow{g} & g(\mathcal{B}_{\text{std}}(X)) & \hookrightarrow & \mathcal{LBS}(X) \\ \downarrow & & \downarrow & \nwarrow & \\ \partial_{\text{LB}}X & \xrightarrow{i_S} & \partial_{\text{LBS}}X & & \end{array}$$

Theorem 5.11. *Suppose X is an $r\text{CAT}(0)$ space. Then the map of a bouquet sequence to itself defines a natural surjection $\sigma : \partial_{\text{BS}}X \rightarrow \partial_{\text{LBS}}X$.*

Proof. The fact that σ is well-defined is trivial. As for surjectivity, suppose that $x = (x_n)$ is a loose δ -bouquet sequence. By taking a subsequence if necessary, we may assume that $d(o, x_{n+1}) \geq d(o, x_n) + 1$ for all n . By the proof of Theorem 5.8, x is the sequence of tips of a loose (δ', D) -bouquet β (for any short function D of our choice, and some δ' dependent only on δ and D). Let $\beta' = (\beta'_n)$ be a bouquet obtained by pruning β , as in the surjectivity part of the proof of Theorem 4.18; we denote by $x' = (x'_n)$ the associated sequence of tips, and let K be the constant of asymptoticity as in Definition 4.8. Now β is loosely asymptotic to β' , so the associated sequences of tips x and x' are loosely asymptotic by the proof that i_S is well-defined in Theorem 5.8. \square

Remark 5.12. We have proved in the above theorem that the following diagram of natural maps commutes, with injections and surjections as indicated. Note that the vertical maps are quotient maps, and that the map from $\mathcal{BS}(X)$ to $\mathcal{LBS}(X)$ is the identity map. Note also that the natural (composition) map from $\mathcal{BS}(X)$ to $\partial_{\text{LBS}}X$ is surjective.

$$\begin{array}{ccc} \mathcal{BS}(X) & \hookrightarrow & \mathcal{LBS}(X) \\ \downarrow & & \downarrow \\ \partial_{\text{BS}}X & \xrightarrow{\sigma} & \partial_{\text{LBS}}X \end{array}$$

The following examples show that, unlike the maps i_L and i_S in Theorems 4.18 and 5.8, the map σ in Theorem 5.11 is not necessarily an injection.

Example 5.13. Consider the complex conjugate sequences $x = (x_n)$ and $y = (y_n)$ in the complex plane \mathbb{C} given by $x_n = 4^n + 2^n i$, $y_n = 4^n - 2^n i$, $n \in \mathbb{N}$, where $i = \sqrt{-1}$. Suppose $m \leq n$, $m, n \in \mathbb{N}$. The simple estimate $\sqrt{1+t} \leq 1+t$ when $t > 0$ yields

$$d(o, x_m) = \sqrt{2^{4m} + 2^{2m}} = 2^{2m} \sqrt{1 + 2^{-2m}} \leq 2^{2m} + 1,$$

and similarly

$$d(x_m, x_n) \leq 2^{2n} - 2^{2m} + 1.$$

Thus

$$2 \langle o, x_n; x_m \rangle \leq (2^{2m} + 1) + (2^{2n} - 2^{2m} + 1) - 2^{2n} = 2,$$

and so x is a bouquet sequence. By symmetry, y is also a bouquet sequence. It is readily verified that x and y are not equivalent as bouquet sequences, but that they are equivalent as loose bouquet sequences.

Example 5.14. Our second example is a variant of Example 5.13 with $x_n^t = 4^n + 2^n ti$, where $-1 \leq t \leq 1$ and again $i = \sqrt{-1}$. For each t , $x^t = (x_n^t)$ is a bouquet sequence in the convex subset X of the complex plane consisting of all $x + yi$ with $0 \leq x < \infty$ and $y^2 \leq x$. If $s \neq t$ then x^s and x^t are inequivalent as bouquet sequences (although they are loosely equivalent). It follows that $\partial_{\text{BS}}X$ has the cardinality of the continuum even though $\partial_{\text{B}}X$ is a singleton set. Thus the map σ is far from being injective in this case.

Although $\partial_{\text{BS}}X$ and $\partial_{\text{LBS}}X$ are in general different, we now show that both can be naturally identified with the Gromov boundary $\partial_{\text{G}}X$ of a Gromov hyperbolic space X .

Theorem 5.15. *Suppose X is an $r\text{CAT}(0)$ space. Then every loose bouquet sequence is a Gromov sequence, and this identity map induces natural maps $\mu : \partial_{\text{LBS}}X \rightarrow \partial_{\text{G}}X$ and $\mu' = \mu \circ \sigma : \partial_{\text{BS}}X \rightarrow \partial_{\text{G}}X$. If X is Gromov hyperbolic, then μ and σ (and hence μ') are bijective.*

Proof. Suppose (x_n) is a (loose) bouquet sequence. By direct computation,

$$(5.16) \quad d(o, x_m) = \langle x_m, x_n; o \rangle + \langle o, x_n; x_m \rangle.$$

Taking $n \geq m$ and letting $m \rightarrow \infty$, the left-hand side of the above equation tends to infinity faster than $\langle o, x_n; x_m \rangle = \delta(d(o, x_m))$, and so $\langle x_m, x_n; o \rangle$ must tend to infinity as $m, n \rightarrow \infty$. Thus (x_n) is a Gromov sequence.

In a similar way, it follows that any pair of (loosely) asymptotic bouquet sequences must be equivalent as Gromov sequences. Thus the identity map gives rise to well-defined natural maps $\mu : \partial_{\text{LBS}}X \rightarrow \partial_{\text{G}}X$ and $\mu' : \partial_{\text{BS}}X \rightarrow \partial_{\text{G}}X$. Since $\sigma : \partial_{\text{BS}}X \rightarrow \partial_{\text{LBS}}X$ is also induced by an identity map on bouquet sequences, it follows that $\mu' = \mu \circ \sigma$. It remains to prove that these maps are bijective under the added assumption that X is Gromov hyperbolic.

We already know that σ is surjective (Theorem 5.11), and we now prove that μ is surjective. Given a Gromov sequence (x_n) , we thin it out if necessary to ensure that

$$\langle x_m, x_n; o \rangle \geq m \wedge n, \quad m, n \in \mathbb{N}.$$

Note that a subsequence of a Gromov sequence is always an equivalent Gromov sequence. In particular, $d(x_n, o) = \langle x_n, x_n; o \rangle \geq n$, $n \in \mathbb{N}$. We may also assume that $d(x_n, o)$ is an increasing function of n . For each $n \in \mathbb{N}$, let $\lambda_n : [0, L_n] \rightarrow X$ be a unit speed 1-short path from o to x_n , let γ_n be the initial segment of λ_n of length n , and let $y_n = \gamma_n(n)$. As before,

$$d(o, y_n) = \langle y_n, x_n; o \rangle + \langle o, x_n; y_n \rangle.$$

Since λ_n is a 1-short segment, we must have $\langle o, x_n; y_n \rangle \leq 1/2$, and so $\langle y_n, x_n; o \rangle \rightarrow \infty$ as $n \rightarrow \infty$. A repeated application of hyperbolicity gives

$$\langle y_m, y_n; o \rangle \geq \langle y_m, x_m; o \rangle \wedge \langle x_m, x_n; o \rangle \wedge \langle x_n, y_n; o \rangle - 2c,$$

so $\langle y_m, y_n; o \rangle \rightarrow \infty$ as $m, n \rightarrow \infty$. Thus (y_n) is a Gromov sequence and it is equivalent to (x_n) .

Suppose now that $m, n \in \mathbb{N}$, $m < n$, and let $z_m = \gamma_n(m)$. Then $d(y_m, o) \leq m \leq \langle x_m, x_n; o \rangle$, so it follows from the Tripod Lemma (Lemma 5.2) that $d(y_m, z_m) \leq 4c + 2$.

Consequently,

$$\begin{aligned} 2 \langle o, y_n; y_m \rangle &= d(o, y_m) + d(y_m, y_n) - d(o, y_n) \\ &\leq m + (4c + 2 + n - m) - (n - 1) = 4c + 3, \end{aligned}$$

and so (y_n) is a bouquet sequence. Thus μ is surjective as required.

To prove injectivity of μ and σ , it suffices to show that if $x = (x_n)$ and $y = (y_n)$ are non-asymptotic c -bouquet sequences, then x and y are not equivalent as Gromov sequences. Since x and y are non-asymptotic, we can fix indices $M, N \in \mathbb{N}$ such that

$$(5.17) \quad J := \langle o, x_M; y_N \rangle \wedge \langle o, y_N; x_M \rangle > 2c + 6\delta.$$

We claim that

$$\langle x_m, y_n; o \rangle \leq (d(o, x_M) \vee d(o, y_N)) + c + 2\delta, \quad \text{for all } m > M, n > N.$$

Assuming this claim, we see that $x' := (x_{i+M})_{i=1}^\infty$ is not equivalent to $y' := (y_{i+N})_{i=1}^\infty$. Since x is equivalent to x' , and y to y' , it follows that x is not equivalent to y , as required.

Let us prove the claim. Suppose $m > M$ and $n > N$. Fixing an arbitrary $h > 0$, we choose a h -short path $\gamma_1 : [0, L_1] \rightarrow X$ from o to x_m parametrized by arclength. By Lemma 5.1, $d(x_M, \gamma_1) \leq c_0 := c + 2\delta + h$, so let $u_1 := \gamma_1(t_1)$, $0 \leq t_1 \leq L_1$, be such that $d(x_M, u_1) \leq c_0$. Similarly, we choose a h -short path $\gamma_2 : [0, L_2] \rightarrow X$ from o to y_n parametrized by arclength, and then there exists $v_2 := \gamma_2(t_2)$, $0 \leq t_2 \leq L_2$, such that $d(y_N, v_2) \leq c_0$. It follows that

$$(5.18) \quad \langle o, u_1; v_2 \rangle \wedge \langle o, v_2; u_1 \rangle > J - 2c_0.$$

Without loss of generality, we assume that $t_1 \leq t_2$. Writing $u_2 = \gamma_2(t_1)$, we see

$$d(u_1, v_2) - d(u_1, u_2) \leq d(u_2, v_2) \leq t_2 - t_1,$$

and, by the shortness of γ_2 ,

$$t_2 - t_1 - h \leq d(o, v_2) - d(o, u_2).$$

It follows that $\langle o, u_2; u_1 \rangle > \langle o, v_2; u_1 \rangle - h/2$, and so (5.18) implies that if $h > 0$ is sufficiently small, then

$$\langle o, u_2; u_1 \rangle > J - 2c_0 - h/2 > 2\delta + h.$$

By h -shortness we have

$$|\langle o, u_2; u_1 \rangle - \langle o, u_1; u_2 \rangle| \leq |d(o, u_1) - d(o, u_2)| \leq h$$

and so again if $h > 0$ is sufficiently small, then

$$\langle o, u_1; u_2 \rangle > J - 2c_0 - 3h/2 > 2\delta + h.$$

Thus for $h > 0$ sufficiently small, we have

$$d(u_1, u_2) = \langle o, u_2; u_1 \rangle + \langle o, u_1; u_2 \rangle > 4\delta + 2h.$$

In view of Lemma 5.2, we conclude that

$$\langle x_m, y_n; o \rangle < d(o, u_1) \leq d(o, x_M) + c_0 = d(o, x_M) + c + 2\delta + h.$$

Since $h > 0$ is arbitrary, the claim follows. \square

The following result shows that the natural maps μ and μ' may fail to be injective if X is not Gromov hyperbolic. These maps can also fail to be surjective in complete CAT(0) spaces according to [4, Theorem 1].

Proposition 5.19. *Suppose first that $X = \mathbb{R}^n$ for $n > 1$ with the Euclidean metric attached. Then $\partial_B X = \partial_{\text{LBS}} X$ has the cardinality of the continuum, while $\partial_G X$ is a singleton set. The natural maps μ and μ' are not injective.*

Proof. Since X is a complete CAT(0) space, $(\partial_I X, \tau_C)$ is homeomorphic to the sphere $S := \partial B(0, 1)$, and so its cardinality is that of the continuum. The same is true of $\partial_B X = \partial_{\text{LBS}} X$ by Theorem 4.20. We claim that $\partial_G X$ is a singleton set. Assuming this claim, it is clear that μ and μ' cannot be injective.

It remains to justify our claim. Certainly $\partial_G X$ is nonempty because of the natural map μ . We now appeal to Theorem 2.2 of [5] which states that μ is surjective if X is a proper geodesic space. In fact, as is clear from the proof of that result, μ is induced by the map that takes a geodesic ray $\gamma : [0, \infty) \rightarrow X$ parametrized by arclength to the Gromov sequence $(\gamma(t_n))_{n=1}^\infty$, where (t_n) is any sequence of non-negative numbers with limit infinity. Since the ideal boundary of a complete CAT(0) space can be viewed as the set of geodesic rays from any fixed origin, it follows that we get representatives of all points in $\partial_G \mathbb{R}^2$ by considering only the Gromov sequences $x^t := (na_t)_{n=1}^\infty$, where $a_t = (\cos t, \sin t) \in \mathbb{R}^2$, $t \in \mathbb{R}$. A straightforward calculation shows that $(x^t, x^s) \in E$ for all pairs t, s , except when $|t - s|$ is an odd multiple of π , i.e. except when x^t and x^s are tending to infinity in opposite directions. But in the exceptional case, we have $(x^t, x^{t+\pi/2}) \in E$ and $(x^{t+\pi/2}, x^s) \in E$, so all Gromov sequences are equivalent, and we have proved our claim. \square

Finally, we relate the Gromov and end boundaries. As in Proposition 5.19, we view all our varieties of the bouquet boundary as being the same. If we do not make this identification, then the second statement of this result should instead state that $\eta = \phi \circ \mu \circ i_S \circ i_L$, where i_L and i_S are as in Theorems 4.18 and 5.8, respectively.

Proposition 5.20. *Suppose X is a metric space. Then there is a natural map ϕ from $\partial_G X$ to $\partial_E X$. Furthermore $\eta = \phi \circ \mu$, with η as in Theorem 4.25 and μ as in Theorem 5.15.*

Proof. Suppose $x = (x_n)$ is a Gromov sequence, and let o be the basepoint for $\partial_G X$ and $\partial_E X$, as usual. Let $f(n)$ be the smallest $k \in \mathbb{N}$ such that $\langle x_i, x_j; o \rangle > n$ for all $i, j \geq k$, so that f is a monotonically increasing sequence with limit infinity (because x is a Gromov sequence). Note in particular that $d(x_i, o) = \langle x_i, x_i; o \rangle > n$ when $i \geq f(n)$.

Let U_n be the component of $X \setminus \overline{B(o, n)}$ containing $x_{f(n)}$. We claim that (U_n) is an end. To show this, it suffices to show that $x_m \in U_n$ for all $m \geq f(n)$. If this were not true, then any path from x_m to $x_{f(n)}$ would have to pass through $B(o, n)$, and so $d(x_m, x_{f(n)})$ would be larger than $(d(x_m, o) - n) + (d(x_{f(n)}, o) - n)$, which would imply that $\langle x_m, x_{f(n)}; o \rangle < n$, in contradiction to our construction. Thus we have a map from Gromov sequences to ends, and we can see in a similar fashion that if x, y are two Gromov sequences with $(x, y) \in E$, then this map takes them to the same end. It follows that this map induces a natural map $\phi : \partial_G X \rightarrow \partial_E X$.

The last statement in the theorem follows easy because η and ϕ are both induced by set containment, and ϕ by the tip map (or just the identity map, if we view the bouquet boundary as being given by the $\partial_{\text{LBS}}X$ variant). \square

Note that the $\phi : \partial_{\text{G}}X \rightarrow \partial_{\text{E}}X$ need not be injective even if X is a complete $\text{CAT}(-1)$ space (and so both $\text{CAT}(0)$ and Gromov hyperbolic), as evidenced by the hyperbolic plane. Also ϕ need not be surjective among complete $\text{CAT}(-1)$ spaces [4, Theorem 2].

In summary, by putting together Corollary 4.19, Theorem 5.8, and Remark 5.12, we see that sets of equivalence classes as listed below lead to seven naturally equivalent notions of boundary at infinity for an $\text{rCAT}(0)$ space X , and we call them all the bouquet boundary. (By “naturally equivalent”, we mean that there is a natural bijection.)

- Asymptotic (bouquets or standard bouquets).
- Loosely asymptotic (loose bouquets, bouquets, or standard bouquets).
- Loosely asymptotic (bouquet sequences or loose bouquet sequences).

Furthermore, we have the following commutative diagram of natural maps between the various types of boundaries that we have considered:

$$\begin{array}{ccccccc}
 & & & & \partial_{\text{BS}}X & & \\
 & & & & \downarrow \sigma & & \\
 \partial_{\text{I}}X & \xrightarrow{i_{\text{I}}} & \partial_{\text{B}}X & \xrightarrow{i_{\text{L}}} & \partial_{\text{LB}}X & \xrightarrow{i_{\text{S}}} & \partial_{\text{LBS}}X \\
 & & \downarrow \eta & & & & \downarrow \mu \\
 & & \partial_{\text{E}}X & \xleftarrow{\phi} & \partial_{\text{G}}X & &
 \end{array}$$

Here i_{I} is injective, or bijective if X is complete $\text{CAT}(0)$ (Theorem 4.20). For the bijections i_{L} and i_{S} , see Theorems 4.18 and 5.8, respectively. σ is surjective, or bijective if X is Gromov hyperbolic, in which case μ is also bijective (Theorems 5.11 and 5.15). Conditions for η to be surjective or injective are given in Theorem 4.25 and Remark 4.27, respectively.

6. THE BOUQUET TOPOLOGY

In this section, we define a bouquet topology τ_{B} on $\overline{X}_{\text{B}} := X \cup \partial_{\text{B}}X$ which makes \overline{X}_{B} into a *bordification* of X , i.e. X with its metric topology is a dense subspace of $(\overline{X}_{\text{B}}, \tau_{\text{B}})$.

Throughout this section, we assume implicitly that X is a C - $\text{rCAT}(0)$ space and $\partial_{\text{B}}X$ is nonempty (and so X is unbounded). The origin $o \in X$ is fixed but arbitrary. In all cases D and L_n are as defined for standard bouquets, i.e. $D(t) := 1/(1 \vee (2t))$ for all $t \geq 0$, and $L_n := (2C + 2)^n$. For convenience, $L_0 := 1$.

In order to proceed, we define a version of X that is similar to the definition of $\partial_{\text{B}}X$, i.e. we view X as a set of equivalence classes of objects vaguely resembling standard bouquets.

Definition 6.1. A *mother bouquet* from $o \in X$ to $x \in X$ is simply a D -short unit speed path $\gamma : [0, L] \rightarrow X$ from o to x . It is convenient to define the associated *finite length bouquet* $\beta := (\beta_n)$ from o to x by $\beta_n = \tilde{\gamma}|_{[0, L_n]}$, where $\tilde{\gamma}$ is defined by $\tilde{\gamma}|_{[0, L]} = \gamma$

and $\tilde{\gamma}(t) = x$ for all $t > L$. We also call β the *child* of γ , and $\beta_n(L_n)$, $n \in \mathbb{N}$, the *tips* of β . We write $L'_n := L_n \wedge L$, so that $\beta_n|_{[0, L'_n]}$ is always a unit speed segment.

Note that if $\gamma : [0, L] \rightarrow X$ is a mother bouquet from o to x , then $L - d(o, x) \leq D(d(0, x)) \leq 1$. Thus $\tilde{\gamma}(t) = x$ for all $t \geq d(o, x) + 1$. We say that all finite length bouquets from o to $x \in X$ are *destination equivalent* and denote this equivalence class by $i(x)$. Thus $i : X \rightarrow i(X)$ is a bijection.

Fixing an rCAT(0) space X , we denote by $\mathcal{GB}(X)$ the set of all *generalized bouquets* from o , meaning the set of all standard and finite length bouquets from o . Identifying X with $i(X)$, we view \bar{X}_B as a set of equivalence classes of generalized bouquets, where these classes are defined using destination equivalence for finite length bouquets and asymptoticity for standard bouquets.

Define the product topological space

$$P := \prod_{n=1}^{\infty} \prod_{0 \leq t \leq L_n} X_n^t$$

where X_n^t is the closed ball of all $x \in X$ such that $d(o, x) \leq L_n + 1$; the t -superscript serves only to distinguish between copies of this ball. Denote by $p_{nt} : P \rightarrow X_n^t$ the associated projection maps.

Because the paths β_n of $\beta \in \mathcal{GB}(X)$ are all of subunit speed, we see that $\beta_n(t) \in X_n^t$ and so β can naturally be viewed as an element of P . Let $q : \mathcal{GB}(X) \rightarrow \bar{X}_B$ be the quotient map consistent with our definition of \bar{X}_B , i.e. $q(\beta) = q(\beta')$ if and only if β, β' are either asymptotic standard bouquets or destination equivalent finite length bouquets. P induces a subspace topology on $\mathcal{GB}(X)$, and \bar{X}_B then receives the quotient topology for q .

$$\begin{array}{ccc} \mathcal{GB}(X) & \hookrightarrow & P \\ q \downarrow & & \downarrow p_{nt} \\ \bar{X}_B & & X_n^t \end{array}$$

Definition 6.2. The *bouquet topology* τ_B on \bar{X}_B is the quotient topology for q . Henceforth, \bar{X}_B and $\partial_B X$ are shorthand for (\bar{X}_B, τ_B) and its subspace $(\partial_B X, (\tau_B)_{\partial_B X})$. We call $(\tau_B)_{\partial_B X}$ the bouquet topology on $\partial_B X$ and also denote it simply as τ_B .

In order to give alternative, more explicit, definitions of the bouquet topology, we first define some sets containing elements of \bar{X}_B that are somehow close to $x \in \partial_B X$. In these definitions, which we use throughout this section, we assume that $r > 0$, $n \in \mathbb{N}$, and $0 \leq t \leq L_n$.

$$S'(x, r; n, t) = \{y \in \bar{X}_B \mid \forall \beta \in q^{-1}(x), \beta' \in q^{-1}(y) : d(p_{nt}(\beta), p_{nt}(\beta')) < r\},$$

$$S(x, r; n, t) = \{y \in \bar{X}_B \mid \exists \beta \in q^{-1}(x), \beta' \in q^{-1}(y) : d(p_{nt}(\beta), p_{nt}(\beta')) < r\},$$

$$S_0(x; n, t) = \{y \in \bar{X}_B \mid \exists \beta \in q^{-1}(x), \beta' \in q^{-1}(y) : p_{nt}(\beta) = p_{nt}(\beta')\},$$

$$I(x; n) = \bigcap_{\substack{1 \leq m \leq n \\ 0 \leq t \leq L_n}} S_0(x; n, t).$$

In Figure 1, we give a rough illustration of what $S(x, r; n, t)$ looks like in one particular instance. Here X is the Euclidean plane, $n = 2$, and $q(\beta) = x$. Let us assume

that $y \in S(x, r; 2, t)$ with $y = q(\beta')$, and that $t \geq r \geq 2$. The shortness parameter of $D_2 := D(L_2)$ is fairly small: certainly $D_2 \leq 1/8$, and D_2 is much smaller than this if C is large, so the constraint $d(\beta_2(t), \beta'_2(t)) < r$ forces $d(o, \beta'_2(t))$ to lie in the interval $[t - r - 2D_2, t]$; in particular, $d(o, y)$ cannot be much less than $t - r$. Thus the distance from the dot representing $\beta_2(t)$ to the boundary arc of $S(x, r; n, t)$ closest to o in the diagram would typically be slightly larger than r . On the other hand, the diameter of $S(x, r; n, t) \cap \{z \in X \mid d(z, o) = t\}$ is typically larger than this since we require only that $d(\beta_2(t), \beta'_2(t)) < r$. By our definition, it is not at first obvious that this puts any upper bound on how far other representatives of $[\beta]$ and $[\beta']$ might be from each other. However it follows from (6.8) below that if we redefined β, β' to be other members of these respective equivalence classes of generalized bouquets, then we would still have $d(\beta_2(t), \beta'_2(t)) < r + 10C + 8$, which justifies the fact that $S(x, r; 2, t)$ is bounded roughly by rays from the origin whose distance from $\beta_2(t)$ is larger than r , but not larger than $r + 10C + 8$.

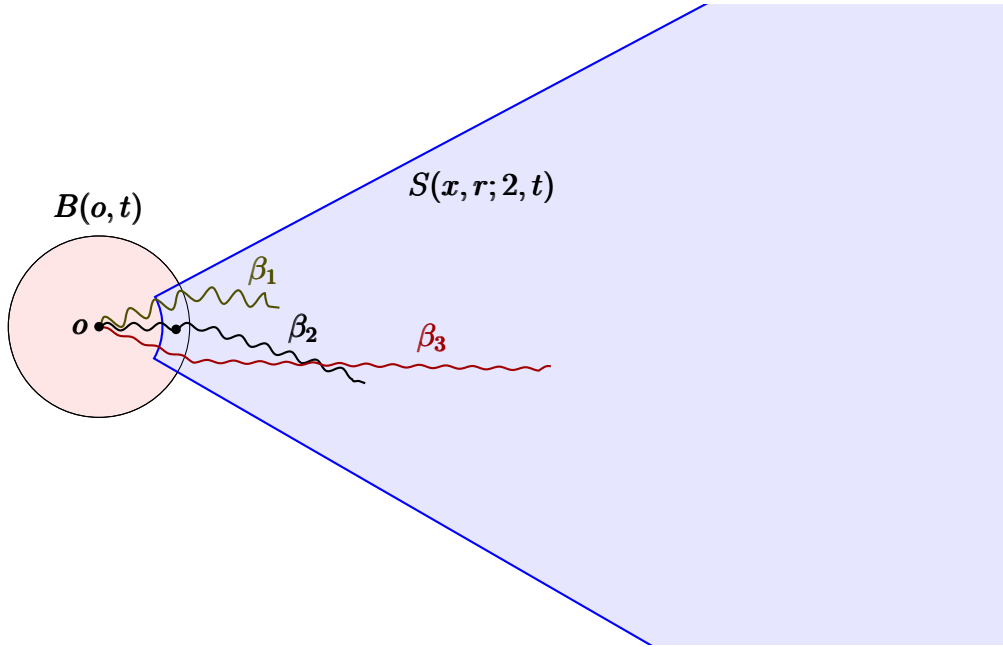


FIGURE 1. A basic neighborhood of $x \in \partial_B X$

Trivially, $S_0(x; n, t) \subset S(x, r; n, t)$ and $S'(x, r; n, t) \subset S(x, r; n, t)$. The next two lemmas together show that containments in the reverse direction (with a change of arguments!) are also possible: these will be crucial to establishing alternative definitions for τ_B .

Lemma 6.3. *For all $R > 0$ and $n_0 \in \mathbb{N}$, there exist $N \in \mathbb{N}$ and $T > 0$ such that $S(x, R; N, T) \subset I(x; n_0)$ for all $x \in \partial_B X$.*

Proof. It suffices to prove the lemma when R is large, so we assume without loss of generality that $R \geq 1$. Let $s := (2R + 1)L_{n_0} + R + 1$. We will show that the lemma is true for the choice of parameters (N, T) as long as N is so large that $L_{N-1} \geq s$, and $T \in [s, L_{N-1}]$. Note that $N > n_0 + 1$ because $L_{N-1} \geq s > L_{n_0}$. Let $\beta^1 = (\beta_n^1)$ be such that $x = q(\beta^1)$.

For $m > n_0$, we apply Lemma 3.6 to the bouquet inequality

$$d(\beta_N^1(L_N \wedge L_m), \beta_m^1(L_N \wedge L_m)) \leq 2C + 2$$

to deduce that

$$(6.4) \quad d(\beta_N^1(t), \beta_m^1(t)) \leq C + 1, \quad 0 \leq t \leq L_{N-1} \wedge L_{m-1}.$$

Suppose now that $y \in S(x, R; N, T)$, and let $\beta^2 = (\beta_n^2)$ be a generalized bouquet such that $y = q(\beta^2)$ and

$$(6.5) \quad d(p_{NT}(\beta^1), p_{NT}(\beta^2)) < R.$$

We claim that $d(\beta_N^1(t), \beta_N^2(t)) \leq C + 1$ for $0 \leq t \leq L_{n_0}$. Assuming this claim and combining it with (6.4), we see that for $m > n_0$ we have

$$d(\beta_m^1(t), \beta_N^2(t)) \leq d(\beta_m^1(t), \beta_N^1(t)) + d(\beta_N^1(t), \beta_N^2(t)) \leq 2C + 2, \quad 0 \leq t \leq L_{n_0}.$$

Thus we can define a new standard bouquet β^3 with $q(\beta^3) = x$ by the equations $\beta_k^3 = \beta_N^2|_{[0, L_k]}$ for all $k \leq n_0$ and $\beta_k^3 = \beta_{k+1}^1|_{[0, L_k]}$ for all $k > n_0$. Since $\beta_n^3 = \beta_n^2$ for all $n \leq n_0$, it follows that $y \in S_0(x; n, t)$ for all $n \leq n_0$, $0 \leq t \leq L_n$, and the theorem follows.

It remains to justify the claim. Because $T \geq RL_{n_0}$, the claim follows by applying Lemma 3.6 to (6.5) if $y \in \partial_B X$. It also follows in the same way if $y \in X$ and $\beta_N^2|_{[0, T]}$ is a unit speed path, i.e. if $L'_N \geq T$, where L'_N is as in Definition 6.1 for the finite length bouquet β^2 .

Suppose therefore that $L'_N < T$. The inequality $d(\beta_N^1(T), \beta_N^2(T)) < R$, the 1-shortness of $\beta_N^1|_{[0, T]}$, and the triangle inequality together imply that $L'_N > T - (R + 1)$, which in turn implies that $L'_N \geq (2R + 1)L_{n_0}$. Now $\beta_N^1|_{[0, L'_N]}$ and $\beta_N^2|_{[0, L'_N]}$ are unit speed paths and $d(\beta_N^1(L'_N), \beta_N^2(L'_N)) < 2R + 1$. Since $T - (R + 1) \geq (2R + 1)L_{n_0}$, the claim follows as before from Lemma 3.6. \square

Remark 6.6. We note two aspects of the proof of Lemma 6.3:

- (a) If (N, T) is one particular choice of data for which the proof works, then it also works for any (N', T') such that $T' \geq T$ and $L_{N'-1} \geq T'$. In particular, T can be taken to be arbitrarily large.
- (b) Suppose $n_0 \in \mathbb{N}$ and $R > 0$ are fixed. For every sequence (N_n) of integers and every unbounded sequence (T_n) such that $0 < T_n \leq L_{N_n-1}$, there exists $n \in \mathbb{N}$ such that $S(x, R; N_n, T_n) \subset I(x; n_0)$ for all $x \in \partial_B X$. For instance, if we write $S(x; n) := S(x, 1; n, L_{n-1})$, then there exists $n \in \mathbb{N}$ such that $S(x; n) \subset I(x; n_0)$.

Lemma 6.7. $S_0(x; n, t) \subset S'(x, 10C + 8; n, t)$ for all $x \in \partial_B X$, $n \in \mathbb{N}$, and $0 \leq t \leq L_n$.

Proof. Suppose $\beta, \beta' \in \mathcal{GB}(X)$ with $z := q(\beta) = q(\beta') \in \overline{X}_B$. The lemma follows immediately once we show that

$$(6.8) \quad d(\beta_n(t), \beta'_m(t)) \leq 5C + 4, \quad 0 \leq t \leq L_n \wedge L_m, \quad n, m \in \mathbb{N}.$$

Suppose first that $z \in X$ and pick $N \in \mathbb{N}$, $N \geq n \vee m$, such that $\beta_N(L_N) = \beta'_N(L_N) = z$. It follows from Lemma 3.9 that $d(\beta_N(t), \beta'_N(t)) \leq C$ for all $0 \leq t \leq L_N$. (Note that we cannot use Lemma 3.6 to get this estimate because β_N and β'_N might not be of equal length.) Thus for $0 \leq t \leq L_n \wedge L_m$ we have

$$\begin{aligned} d(\beta_n(t), \beta'_m(t)) &\leq d(\beta_n(t), \beta_N(t)) + d(\beta_N(t), \beta'_N(t)) + d(\beta'_N(t), \beta'_m(t)) \\ &\leq (2C + 2) + C + (2C + 2) = 5C + 4. \end{aligned}$$

For $z \in \partial_B X$, we can similarly deduce (6.8) from the limiting estimate

$$\limsup_{N \rightarrow \infty} d(\beta_N(t), \beta'_N(t)) \leq C, \quad 0 \leq t \leq L_n \wedge L_m.$$

This last estimate follows from Lemma 3.6 with data $a_1 = a_2 = o$, $b_1 = \beta_N(L_N)$, $b_2 = \beta'_N(L_N)$, because of the uniformly boundedness of $d(\beta_N(L_N), \beta'_N(L_N))$ and the fact that $(L_n \wedge L_m)/L_N \rightarrow 0$ as $N \rightarrow \infty$. \square

In preparation for the next theorem, let us define some sets associated with any choice of $x \in \overline{X}_B$ and $R \geq 0$. For $x \in \partial_B X$, let

$$\begin{aligned} \mathcal{B}_0(x) &= \{S_0(x; n, t) \mid n \in \mathbb{N}, 0 < t \leq L_n\}, \\ \mathcal{B}_{1,R}(x) &= \{S(x, r; n, t) \mid r > R, n \in \mathbb{N}, 0 < t \leq L_n\}, \\ \mathcal{B}_{2,R}(x) &= \{S'(x, r; n, t) \mid r > R, n \in \mathbb{N}, 0 < t \leq L_n\}, \end{aligned}$$

while for $x \in X$, we simply define

$$\mathcal{B}_0(x) = \mathcal{B}_{1,R}(x) = \mathcal{B}_{2,R}(x) = \{B(x, r) \mid r > 0\}.$$

Theorem 6.9. *Suppose X is C -rCAT(0). Then*

- (a) *For each $R \geq 0$, $\mathcal{B}_{1,R}(x)$ is a neighborhood basis for (\overline{X}_B, τ_B) at $x \in \overline{X}_B$, all of whose elements are open.*
- (b) *$\mathcal{B}_0(x)$ is a neighborhood basis at $x \in \overline{X}_B$ for (\overline{X}_B, τ_B) .*
- (c) *For each $R \geq 10C + 8$, $\mathcal{B}_{2,R}(x)$ is a neighborhood basis at $x \in \overline{X}_B$ for (\overline{X}_B, τ_B) .*

Also \overline{X}_B is a first countable bordification of X .

Proof. By definition,

$$\mathcal{B}(x) := \{S(x, r; n, t) \mid r > 0, n \in \mathbb{N}, 0 < t \leq L_n\}$$

forms a neighborhood sub-basis for \overline{X}_B at each $x \in \overline{X}_B$, and all elements of $\mathcal{B}(x)$ are open. When $x \in \partial_B X$, $\mathcal{B}(x) = \mathcal{B}_{1,0}(x)$, so this is a neighborhood sub-basis, and in fact a neighborhood basis by Lemma 6.3. Applying Lemma 6.3 again it is readily deduced that for all $R > 0$, $\mathcal{B}_{1,R}(x)$ and $\mathcal{B}_0(x)$ are neighborhood bases at $x \in \partial_B X$. Lemma 6.7 then implies that $\mathcal{B}_{2,R}(x)$ is a neighborhood basis at $x \in \partial_B X$ whenever $R \geq 10C + 8$.

Suppose instead that $x \in X$. We must show that $\mathcal{B}_0(x) = \{B(x, r) \mid r > 0\}$ is a basis of open neighborhoods at $x \in X$ for τ_B , or equivalently that $(\tau_B)_X$ coincides with the metric topology τ . From the form of $\mathcal{B}_0(x)$, it clearly suffices to show that it is a neighborhood sub-basis at $x \in X$ for τ_B .

We claim that any ball $B(x, r)$ equals $S(x, r; n, t)$ for any choice of $t > d(o, x) + r + 1$ and n such that $L_n \geq t$. Since $\mathcal{B}(x)$ is a neighborhood sub-basis for τ_B at $x \in X$, it follows from this claim that $(\tau_B)_X$ is at least as fine as τ . To justify our claim, we suppose that $y \in S(x, r; n, t)$ with $x = q(\beta)$, $y = q(\beta')$, and $d(p_{nt}(\beta), p_{nt}(\beta')) < r$. Because $t > d(o, x) + 1$, we must have $p_{nt}(\beta) = x$. Thus

$$d(o, p_{nt}(\beta')) \leq d(o, x) + d(x, p_{nt}(\beta')) < d(o, x) + r.$$

Since $\beta'_n|_{[0, L'_n]}$ is 1-short and $t > (d(o, x) + r) + 1$, we must have $p_{nt}(\beta') = y$, and so $d(x, y) < r$. Thus $S(x, r; n, t) \subset B(x, r)$. The reverse containment is proved in a similar fashion.

To prove that conversely τ is at least as fine as $(\tau_B)_X$, we show that for fixed but arbitrary $x \in X$, $0 < r < 1$, $n \in \mathbb{N}$, and $0 < t \leq L_n$, $S(x, r; n, t)$ contains some ball

$B(x, \delta)$. First pick a $(D/2)$ -short unit speed path of length L from o to x , and then let $\beta = (\beta_k) \in \mathcal{GB}(X)$ be its child. Also let $\delta := r \wedge [D(d(o, x))/4]$. For $y \in B(x, \delta)$, we pick a unit speed path $\lambda_y : [0, l_y] \rightarrow X$ from x to y , with $l_y < \delta$, and then define $\gamma^y : [0, L + l_y] \rightarrow X$ by the formula

$$\gamma^y(s) = \begin{cases} \gamma(s), & 0 \leq s \leq L, \\ \lambda_y(s - L), & L < s \leq L + l_y. \end{cases}$$

Using the 1-Lipschitz property of D , it is readily verified that γ^y is D -short, and so γ^y is a mother bouquet from o to y . By construction it is clear that $d(\gamma(s), \gamma^y(s)) < r$ for all $s > 0$. In particular this last inequality holds for $s = t$, and so $B(x, \delta) \subset S(x, r; n, t)$.

As for first countability, it is clear that there exists a countable neighborhood base at each $x \in X$, and a countable neighborhood base at $x \in \partial_B X$ is given by Remark 6.6(b). To see that \overline{X}_B is a bordification of X , we need to show that the basic neighborhood $S_0(x; n, t)$ of $x \in \partial_B X$ always contains a point of X . But this is easy since $\beta_n(L_n) \in S_0(x; n, t) \cap X$ whenever $\beta = (\beta_n) \in q^{-1}(x)$. \square

We already know that \overline{X}_B as a set is independent of the basepoint o (Corollary 4.17). We now show that the associated topology is also independent of o .

Theorem 6.10. *The topology τ_B is independent of the basepoint o .*

Proof. Suppose $o, o' \in X$ are two basepoints in X . In view of the definition of the neighborhood bases given in Theorem 6.9, it suffices to show that the topology with respect to these two basepoints is the same in the vicinity of each $x \in \partial_B X$. By symmetry of o, o' , it therefore suffices to exhibit a neighborhood basis at x with basepoint o' such that every element of this basis contains some neighborhood of x with respect to the basepoint o . To facilitate this comparison, we add $\omega \in \{o, o'\}$ as a superscript to our notation, writing τ_B^ω , $S^\omega(x, r; n, t)$, $\mathcal{B}_{1,R}^\omega(x)$, etc. In view of Theorem 4.14, we can write $\partial_B X$ and \overline{X}_B without such a superscript; however we should write the quotient maps as $q^\omega : \mathcal{GB}^\omega(X) \rightarrow \overline{X}_B$.

By Theorem 6.9, $\mathcal{B}_{1,R}^{o'}(x)$ is neighborhood basis for $(\overline{X}_B, \tau_B^{o'})$ for any given $R \geq 0$. Choosing $R := 6C + 4 + 2d(o, o')$, a general element of $\mathcal{B}_{1,R}^{o'}(x)$ has the form $S^{o'}(x, r'; n, t)$ for some $r' > R$, $n \in \mathbb{N}$, and $0 < t \leq L_n$. We claim that such a general basis element contains the neighborhood $S^o(x, r; n, t)$, where $r = r' - R$.

Suppose $y \in S^o(x, r; n, t)$ for some $r > 0$, $n \in \mathbb{N}$, $0 < t \leq L_n$. Thus for $z \in \{x, y\}$, there are generalized bouquets $\beta^{z,1} = (\beta_m^{z,1})_{m=1}^\infty$ from o such that $q^o(\beta^{z,1}) = z$ and $d(\beta_n^{x,1}(t), \beta_n^{y,1}(t)) < r$.

Let us first examine the construction in the proof of Theorem 4.14, where $c = c' = 2C + 2$ and we reserve r, n, t to have their specific meanings in the context of $S^o(x, r; n, t)$. Stripped of its fine details, the construction of a standard bouquet $\beta^{x,2} = (\beta_m^{x,2})_{m=1}^\infty$ from o' that is asymptotic to $\beta^{x,1}$ in the proof of Theorem 4.14 is as follows: first we take a sequence of sufficiently short paths whose m th entry is a path from o' to $x_m := \beta_m^{x,1}(L_m)$ (we can use x_m as the final point rather than some intermediate point y_m as in the original proof because in this section $D(t) := 1/(1 \vee (2t))$: see Remark 4.16), then we take a subsequence of this sequence, and finally we suitably prune this subsequence. In particular there exists $N \geq n$ such that $\beta_n^{x,2}$ is an initial segment of a sufficiently short path $\lambda_N : [0, M_N] \rightarrow X$ from o' to x_N that is parametrized by arclength. Writing $u = \beta_n^{x,1}(t)$, $v = \beta_N^{x,1}(t)$, we have $d(u, v) \leq 2C + 2$

by one of the defining conditions for standard bouquets. Next letting $w := \beta_N^{x,2}(t)$, it follows from Remark 3.8 that $d(v, w) \leq C + d(o, o')$, and so $d(u, w) \leq 3C + 2 + d(o, o')$.

Suppose $y \in \partial_B X$. Applying the argument of the previous paragraph to y in place of x , we get a standard bouquet $\beta^{y,2} = (\beta_m^{y,2})_{m=1}^\infty$ from o' that is asymptotic to $\beta^{y,1}$ such that $d(u', w') \leq 3C + 2 + d(o, o')$ where $u' = \beta_n^{y,1}(t)$ and $w' := \beta_N^{y,2}(t)$. Since $d(u, u') < r$, we conclude that

$$d(w, w') \leq d(w, u) + d(u, u') + d(u', w') < r + R,$$

and so $y \in S^{o'}(x, r + R; n, t)$ as claimed.

Suppose instead that $y \in X$. Let $\beta^{y,2}$ be the child of a mother bouquet from o' to y of length L , let L' be length of $\beta_n^{y,1}$, and let $u' = \beta_n^{y,1}(t)$, $w' := \beta_N^{y,2}(t)$. By Remark 3.8, we see that $d(u', w') \leq C + d(o, o')$ if $t < L \wedge L'$, and otherwise shortness gives $d(u', w') \leq 1 + d(o, o')$. As before

$$d(w, w') \leq d(w, u) + d(u, u') + d(u', w') < r + 4C + 3 + 2d(o, o') < r + R,$$

and so again $y \in S^{o'}(x, r + R; n, t)$. Thus our claim follows and the proof is done. \square

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. First countability was proved in Theorem 6.9. To prove that \overline{X}_B is Hausdorff, we suppose $x, y \in \overline{X}_B$ are distinct. If one or both of x, y lie in X , then Theorem 6.9 implies that they have disjoint neighborhoods: for instance if $x \in \partial_B X$ and $y \in X$ then $B(y, 1)$ and $S(x, 1, n, t)$ are disjoint whenever $t > |y| + 2$ and $L_n \geq t$ (as in the proof that $(\tau_B)_X$ is at least as fine as τ in the proof of Theorem 6.9). It therefore suffices to consider the case where $x, y \in \partial_B X$, and so $x = q(\beta^x)$, $y = q(\beta^y)$, where β^x, β^y are non-asymptotic standard bouquets from o . Since β^x, β^y are not asymptotic, we can find $n \in \mathbb{N}$ so large that $d(\beta_n^x(L_n), \beta_n^y(L_n)) \geq 15C + 14$. Letting $U = S(x, 1; n, L_n)$ and $V = S(y, 1; n, L_n)$, it follows readily from (6.8) that U and V are disjoint neighborhoods of x and y in \overline{X}_B , and so \overline{X}_B is Hausdorff.

We claim that $\mathcal{GB}(X)$ is a closed subset of P . Convergence in the product space P corresponds to pointwise convergence in $\mathcal{GB}(X)$ (meaning convergence for each choice of n, t), so justifying this claim requires us to show that a pointwise limit of a sequence of generalized bouquets is a generalized bouquet. The important step is to note that if for some fixed $n \in \mathbb{N}$ and all $m \in \mathbb{N}$, β_{nm} is a path of subunit speed and length at most L_n from o to x_m (where L_n is defined as always for generalized bouquets), and if $\beta_{nm}(t)$ is pointwise convergent for all $0 \leq t \leq L_n$, then each of these paths lies in the metric space X , so we may apply the Arzelà-Ascoli theorem to deduce that β_{nm} converge uniformly to some limiting path $\beta_n : [0, L_n] \rightarrow X$ of subunit speed. Since the short function D is continuous, β_n is D -short if each β_{nm} is D -short. It readily follows that a pointwise convergent sequence of mother bouquets converges uniformly to a mother bouquet, and that a pointwise convergent sequence of standard bouquets converges to a standard bouquet. The claim follows.

Suppose next that X is proper. Then P is a product of compact spaces and so compact. Compactness is inherited by closed subspaces and by quotients so, applying the above claim, we see that \overline{X}_B is compact. Using Theorem 6.9, we see that $X = \bigcup_{x \in X} B(x, 1)$ is open in \overline{X}_B , and so $\partial_B X$ is closed in \overline{X}_B . Thus $\partial_B X$ is also compact. \square

Proof of Theorem 1.1. By Proposition 3.3, X is C -rCAT(0) for $C := 2 + \sqrt{3}$. By Theorem 4.20, we can identify \overline{X}_I and \overline{X}_B as sets. The neighborhood bases for the bouquet topology τ_B given in Theorem 6.9 and for the cone topology τ_C in Definition 2.1 coincide at each $x \in X$, so it suffices to consider the two neighborhood bases at points $x \in \partial_I X = \partial_B X$.

According to Theorem 4.24, there exists a (unique) unit speed geodesic ray $\gamma^z : [0, \infty) \rightarrow X$ from o that is asymptotic to any given standard bouquet $\beta \in q^{-1}(z)$. We view γ^z as an element of $\mathcal{GB}(X)$ by identifying it with the standard bouquet $\beta^z = (\beta_n^z)$, where $\beta_n^z := \gamma^z|_{[0, L_n]}$. For $z \in X$, let γ^z be the unique unit speed geodesic segment $\gamma^z : [0, d(o, z)] \rightarrow X$ from o to z , and identify γ^z with its child $\beta^z = (\beta_n^z)$. In this way the set of these (unique) unit speed segments or rays from o to all $z \in \overline{X}_B$ is identified with a subset $\mathcal{GB}^*(X)$ of $\mathcal{GB}(X)$ and $q' := q|_{\mathcal{GB}^*(X)} : \mathcal{GB}^*(X) \rightarrow \overline{X}_B$ is bijective, so we identify $\overline{X}_B = \overline{X}_I$ with $\mathcal{GB}^*(X)$. Note that $p_{nt}((q')^{-1}(z))$ is independent of n : in fact it equals $\gamma^z(t)$ (or simply z if $z \in X$ and $t > d(o, z)$).

Viewing $\partial_I X$ in this manner, it follows from Definition 2.1 that the basic neighborhood $U(x, r, t)$ for the cone topology τ_C at $x \in \partial_I X$ is contained in $S(x, r; n, t) \in \mathcal{B}_{1,0}(x)$. On the other hand, it follows from (6.8) that $S(x, r; n, t) \subset U(x, r + 10C + 8, t)$. But the collection of sets $U(x, r', t)$ for all $t > 0$ and $r' > 10C + 8$ forms an open basis for τ_C at x : this follows readily from the containment

$$U(x, 10C + 9, t(10C + 9)/r) \subset U(x, r, t), \quad 0 < r < 1, \quad 0 < t,$$

which in turn follows from the CAT(0) condition. \square

Proof of Theorem 1.2. It suffices to compare the neighborhood bases at $x \in \partial_G X$. We assume that $\delta > 0$ is such that X is δ -hyperbolic, and so X is also C -rCAT(0) for $C := 2 + 4\delta$ by Proposition 3.4. The identification of a standard bouquet with the Gromov sequence of its tips induces an identification of \overline{X}_B and \overline{X}_G as sets; see Remark 5.10 and Theorem 5.15.

We take as a τ_G -neighborhood basis at x the standard one given by Definition 2.3, namely $\{V(x, R) \mid R > 0\}$. Fixing R , we claim that

$$S(x, 1; N, t) \subset V(x, R) \quad \text{whenever } N \in \mathbb{N}, t > 0 \text{ are so large that}$$

$$L_N \geq t > R + (4C + 7)/2.$$

To prove this claim, we assume that $y \in S(x, 1; N, t)$ for such a choice of N and t , and separately show that $y \in V(x, R)$ when $y \in \partial_B X$ and when $y \in X$.

Suppose first that $y \in \partial_B X$. We take the Gromov sequences of tips (x_n) of $\beta^x = (\beta_n^x)$, and (y_n) of $\beta^y = (\beta_n^y)$, where β^x, β^y are standard bouquets such that $q(\beta^x) = x$, $q(\beta^y) = y$, and $d(p_{Nt}(\beta^x), p_{Nt}(\beta^y)) < 1$. For every $m, n \geq N$, we have

$$\begin{aligned} d(x_m, y_n) &= d(\beta_m^x(L_m), \beta_n^y(L_n)) \\ &\leq d(\beta_m^x(L_m), \beta_m^x(t)) + d(\beta_m^x(t), \beta_N^x(t)) + \\ &\quad + d(\beta_N^x(t), \beta_N^y(t)) + d(\beta_N^y(t), \beta_n^y(t)) + d(\beta_n^y(t), \beta_n^y(L_n)) \\ &\leq (L_m - t) + (2C + 2) + 1 + (2C + 2) + (L_n - t) \\ &= (L_m + L_n - 2t) + 4C + 5. \end{aligned}$$

But $d(o, x_m) \geq L_m - 1$ and $d(o, y_n) \geq L_n - 1$, so

$$\langle x_m, y_n; o \rangle \geq t - (4C + 7)/2 > R.$$

Thus $S(x, 1; N, t) \cap \partial_B X \subset V(x, R)$.

The proof for $y \in X$ is mostly similar, so we mention only the differences. First, let $y_n = \beta_n^y(L_n)$, where β^y is a finite length bouquet from o to y . Because $d(\beta_N^x(t), \beta_N^y) < 1$ and β_N^x is 1-short, it follows that $L'_n \geq t - 2$ for $n \geq N$. Thus either $L'_n \geq t$ and we deduce as before that $d(x_m, y_n) \leq (L_m + L'_n - 2t) + 4C + 5$, or $L'_n < t$. In the latter case we have $\beta_n^y(t) = \beta_n^y(L'_n) = y$ for $n \geq N$ and so

$$\begin{aligned} d(x_m, y_n) &= d(\beta_m^x(L_m), \beta_n^y(L'_n)) \\ &\leq d(\beta_m^x(L_m), \beta_m^x(t)) + d(\beta_m^x(t), \beta_N^x(t)) + \\ &\quad + d(\beta_N^x(t), \beta_N^y(t)) + d(\beta_N^y(t), \beta_n^y(L'_n)) \\ &\leq (L_m - t) + (2C + 2) + 1 + 0 \\ &= (L_m - t) + 2C + 3 \leq (L_m + L'_n - 2t + 2) + 2C + 3 \end{aligned}$$

and so we again have $d(x_m, y_n) \leq (L_m + L'_n - 2t) + 4C + 5$. Since also $d(o, y_n) \geq L'_n - 1$, we deduce as before that

$$\langle x_m, y_n; o \rangle \geq t - (4C + 7)/2 > R.$$

But $y_n = y$ for all sufficiently large n , so

$$\langle x_m, y; o \rangle \geq t - (4C + 7)/2 > R.$$

Hence $S(x, 1; N, t) \cap X \subset V(x, R)$. Thus the claim follows and so τ_B is finer than τ_G .

It remains to show conversely that τ_G is finer than τ_B . To show this, we take $\{S(x, r; n, L_{n-1}) \mid n \in \mathbb{N}, r > 4\delta + 2\}$ as a neighborhood basis for τ_B at $x \in \partial_B X$; see Theorem 6.9 and Remark 6.6(b). It suffices to show that

$$V(x, R) \subset S(x, r; n, L_{n-1}), \quad n \in \mathbb{N}, R \geq L_{n-1}, r > 4\delta + 2.$$

Consider first $y \in V(x, R) \cap \partial_B X$. By eliminating some initial elements if necessary from the sequences given by Definition 2.3, we may assume that (a_j) and (b_j) are Gromov sequences such that $[(a_j)] = x$, $[(b_j)] = y$, $\langle a_j, b_k; o \rangle \geq R$ for all $j, k \in \mathbb{N}$, and

$$\langle a_j, a_k; o \rangle \wedge \langle b_j, b_k; o \rangle \geq L_j \wedge L_k, \quad j, k \in \mathbb{N}.$$

The last inequality implies in particular that $d(o, a_j) \wedge d(o, b_j) \geq L_j$, $j \in \mathbb{N}$.

For each $j \in \mathbb{N}$, let β_j^x, β_j^y be the initial segments of length L_j of D -short paths from o to a_j, b_j , respectively, and let $a'_j = \beta_j^x(L_j)$, $b'_j = \beta_j^y(L_j)$ be the associated tips. The Tripod Lemma (Lemma 5.2) tells us that β_j^x and β_j^y are $(4\delta + 2, D)$ -bouquets and that $d(\beta_j^x(s), \beta_k^y(s)) \leq 4\delta + 2$ for all $j, k \in \mathbb{N}$, $s \leq R \wedge L_j \wedge L_k$. This last estimate remains true after taking pruned subsequences of (β_j^x) and (β_j^y) , which we do if necessary in order to get standard bouquets. We may thus assume without loss of generality that β^x, β^y are standard bouquets satisfying $q(\beta^x) = x$, $q(\beta^y) = y$, and $d(\beta_j^x(s), \beta_k^y(s)) \leq 4\delta + 2$ for all $s \in [0, R \wedge L_j \wedge L_k]$ and $j, k \in \mathbb{N}$. In particular, $d(\beta_n^x(t), \beta_n^y(t)) \leq 4\delta + 2$ for $t = L_{n-1}$, and so $y \in S(x, r; n, L_{n-1})$ for every $r > 4\delta + 2$.

The analysis for $y \in V(x, R) \cap X$ is fairly similar. First we choose a Gromov sequence (a_n) such that $[(a_j)] = x$, $\langle a_j, y; o \rangle \geq R$, and $\langle a_j, a_k; o \rangle \geq L_j \wedge L_k$ for all $j, k \in \mathbb{N}$. In particular, $d(o, y) \geq R$ and $d(o, a_j) \geq L_j$ for all $j \in \mathbb{N}$. Define (β_j^x) as before, and let $\beta^y = (\beta_j^y)_{j=1}^\infty$ be a finite length bouquet from o to y . Then (β_x^n) is a $(4\delta + 2, D)$ -bouquet and $d(\beta_j^x(s), \beta_k^y(s)) \leq 4\delta + 2$ for all $s \in [0, R \wedge L_j \wedge L_k]$. In particular, $y \in S(x, r; n, L_{n-1})$ for every $r > 4\delta + 2$. \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, NUI MAYNOOTH, MAYNOOTH, CO. KILDARE, IRELAND

E-mail address: `stephen.buckley@maths.nuim.ie`

UNIVERSITÄT BREMEN, FB 3 - MATHEMATIK, BIBLIOTHEKSTRASSE 1, 28359 BREMEN, GERMANY

E-mail address: `khf@math.uni-bremen.de`